TIME-VARYING FILTER STABILITY AND STATE MATRIX PRODUCTS

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ABSTRACT

We show a new sufficient criterion for time-varying digital filter stability: that the matrix norm of the product of state matrices over a certain finite number of time steps is bounded by 1. This extends Laroche's Criterion 1, which only considered one time step, while hinting at extensions to two time steps. Further extending these results, we also show that there is no intrinsic requirement that filter coefficients be frozen over any time scale, and extend to any dimension a helpful theorem that allows us to avoid explicitly performing eigen- or singular value decompositions in studying the matrix norm. We give a number of case studies on filters known to be time-varying stable, that cannot be proven time-varying stable with the original criterion, where the new criterion succeeds.

1. INTRODUCTION

Stability is an essential aspect of filters. For linear time-invariant (LTI) filters, stability is typically proven by looking at pole locations (all inside unit circle), eigenvalues of the state matrix (magnitudes less than one), or via the Bounded-Input, Bounded-Output (BIBO) concept (bounds on absolute value of impulse response).

Filters used in audio often must vary over time: Some examples are speech synthesis filters, musical filters used in synthesizers (commonly swept), and the time-variation used to improve sound quality of reverbs. Time-varying filters outside of audio include adaptive filters used in medical, seismic, and communications signal processing. For all of these, LTI analysis techniques are not sufficient to prove time-varying stability. For specific filter types, time-varying stability constraints exist, e.g., allpass filters can be stabilized by ensuring they are energy-preserving in time-varying conditions [1, 2, 3]. However, for most filters (e.g., low-pass, high-pass, band-pass), no such simple criteria exists. This motivates us to study the time-varying BIBO stability of filter structures in general, rather than just allpass filters.

Although it is well-known that specific filter realizations such as power-normalized ladder filters [4, 5] remain stable under time-varying coefficients, so long as each time step itself represents an LTI stable filter, power-normalized ladder filters may be computationally expensive due to their large number of multipliers, involve complex coefficient update equations, or simply have different time-varying behavior than desired. This motivates us to study the time-varying BIBO stability of general filter structures, rather than restricting ourselves to power-normalized ladder filters.

In this paper, we extend a classic result by Laroche [6], which he calls *Criterion 1*, which shows that a sufficient but not necessary condition for BIBO stability of a time-varying filter expressed in state-space form is a bound below unity of the state transition matrix norm, plus any finite bound on the input, output, and feedthrough matrices. Laroche also proposes a *Criterion 2*, which uses a similarity transform, and which has been useful when Criterion 1 fails. Extending Criterion 2 is beyond the scope of this paper, so we will not consider it further.

We extend Laroche's Criterion 1 in a few ways: *a)* formalize his results over multiple time steps, *b)* show that this doesn't require frozen filter coefficients, *c)* formalize his lemma about the trace and determinant for multiple time steps, and *d)* develop intuition about when to use multiple time steps.

In the rest of this paper, we review Laroche's result (§2), demonstrate some limitations of this result (§3), present an extension to multiple time steps (§4, our main result), give a theorem which increases the usefulness of this result (§5), and provide several case studies on known time-varying filter types (§6). §7 concludes.

2. REVIEW OF LAROCHE'S RESULTS

In filter design, it is often important to consider bounded-input, bounded-output (BIBO) stability. For a time-varying filter with an impulse response $h[n,i]$, n being the time index and i being the time index of the exciting impulse, a necessary and sufficient condition for BIBO stability is [6]

$$\text{BIBO Stable} \iff \exists G, \forall n, \quad h_\infty[n] = \sum_{i=-\infty}^{\infty} |h[n,i]| < G. \tag{1}$$

A filter realization with inputs $\mathbf{u} \in \mathbb{R}^{N_\text{in}}$, states $\mathbf{x} \in \mathbb{R}^{N}$, and output $\mathbf{y} \in \mathbb{R}^{N_\text{out}}$ can be written in discrete-time state-space form

$$\left\{ \begin{array}{ll} \mathbf{x}[n+1] & = \mathbf{A}[n]\mathbf{x}[n] + \mathbf{B}[n]\mathbf{u}[n] \\ \mathbf{y}[n] & = \mathbf{C}[n]\mathbf{x}[n] + \mathbf{D}[n]\mathbf{u}[n] \end{array} \right. , \tag{2}$$

where $\mathbf{A}[n] \in \mathbb{R}^{N \times N}$ is the state matrix, $\mathbf{B}[n] \in \mathbb{R}^{N \times N_\text{in}}$ is the input matrix, $\mathbf{C}[n] \in \mathbb{R}^{N_\text{out} \times N}$ is the output matrix, and $\mathbf{D}[n] \in \mathbb{R}^{N_\text{out} \times N_\text{in}}$ is the feedthrough matrix. Sometimes it is helpful [7] to study the four matrices together as a system matrix $\mathbf{V} = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$. We will assume a single-input, single-output (SISO) filter, one with $N_\text{in} = N_\text{out} = 1$. Systems written in this form have a causal impulse response, i.e.,

$$h_\infty[n] = \sum_{i=-\infty}^{\infty} |h[n,i]| = \sum_{i=-\infty}^{n} |h[n,i]|. \tag{3}$$

The impulse response is given by

$$h[n,i] = \begin{cases} 0\,, & n < i \\ D[n]\,, & n = i \\ \mathbf{C}[n] \left(\prod_{\ell=i+1}^{n-1} \mathbf{A}[\ell] \right) \mathbf{B}[i]\,, & n > i. \end{cases} \tag{4}$$

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[*] This research was conducted during the 1st author's employment at iZotope, Inc., Boston, MA.

When we refer to products of matrices indexed by time, time increases from right to left. For instance, in the above equation,

$$\prod_{\ell=i+1}^{n-1} \mathbf{A}[\ell] = \mathbf{A}[n-1]\mathbf{A}[n-2]\cdots\mathbf{A}[i+2]\mathbf{A}[i+1]. \quad (5)$$

Combining (1) and (4) gives a necessary and sufficient condition for BIBO stability in terms of the impulse response.

Lemma 2.1. *BIBO stability can be shown by any finite bound on the sum of the closed-form expression for the impulse response:*

$$\text{BIBO Stable} \iff \exists G, \forall n,$$

$$h_\infty[n] = |D[n]| + \sum_{i=n-1}^{-\infty}\left|\mathbf{C}[n]\left(\prod_{\ell=i+1}^{n-1}\mathbf{A}[\ell]\right)\mathbf{B}[i]\right| < G.$$

Laroche gives a sufficient but not necessary condition [6], which he call *Criterion 1*. We will call it *Criterion 1(1)*.

Theorem 2.2 (Criterion 1(1))**.** *Bounding* $\|\mathbf{A}[n]\| < G_A < 1$ *along with any finite bounds on* $\|\mathbf{B}[n]\|$, $\|\mathbf{C}[n]\|$, $\|\mathbf{D}[n]\|$ *for all* n *is a sufficient but not necessary proof of BIBO stability:*

$$\left.\begin{array}{lll}\exists G_A,\ 0\le G_A<1, & \forall n, & \|\mathbf{A}[n]\|\le G_A\\ \exists G_B,\ 0\le G_B, & \forall n, & \|\mathbf{B}[n]\|\le G_B\\ \exists G_C,\ 0\le G_C, & \forall n, & \|\mathbf{C}[n]\|\le G_C\\ \exists G_D,\ 0\le G_D, & \forall n, & \|\mathbf{D}[n]\|\le G_D\end{array}\right\}\implies \begin{array}{l}\text{BIBO}\\ \text{Stable}\end{array}.$$

Proof. A proof is given in [6], and can be taken as a specific case of the general proof given later on in our Theorem (4.1). ■

The last three are satisfied, so long as **B**, **C**, **D** never have any infinite elements, while the first is trickier and will be the focus of all effort in proving stability. Throughout this paper, $\|\cdot\|$ denotes the matrix norm induced by the standard Euclidean vector norm $\|\cdot\|$, which can be thought of in several ways:

Lemma 2.3. *The following three statements are equivalent.*

a) $\|\mathbf{P}\|$ *is equal to the largest singular value* σ *of* $\mathbf{P}$.

b) $\|\mathbf{P}\|$ *is equal to the positive square root of the largest eigenvalue* λ *of* $\mathbf{P}^{\mathrm{T}}\mathbf{P}$.

c) $\|\mathbf{P}\|$ *is the maximum maximum amplification that matrix* $\mathbf{P}$ *can bring to an unit-length vector, i.e,*

$$\|\mathbf{P}\| = \max_{1\le i\le N}[\sigma_i] = \max_{1\le i\le N}\left[\sqrt{\lambda_i}\right] = \max_{\|\mathbf{x}\|=1}\|\mathbf{P}\mathbf{x}\|. \quad (6)$$

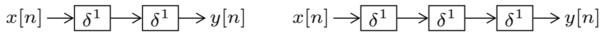
Showing any of these expressions is below unity proves $\|\mathbf{P}\| < 1$.

In a case study, Laroche uses a modified first condition:

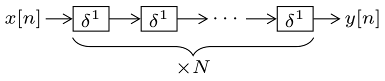
Theorem 2.4 (Criterion 1(2, frozen))**.** *The bound* $\|\mathbf{A}[n]^2\| < G_A < 1$, *along with any finite bounds on* $\|\mathbf{B}[n]\|$, $\|\mathbf{C}[n]\|$, $\|\mathbf{D}[n]\|$ *for all* n *is a sufficient but not necessary proof of BIBO stability, for filters whose coefficients change no more often than every other sample:*

$$\left.\begin{array}{lll}\exists G_A,\ 0\le G_A<1, & \forall n, & \|\mathbf{A}[n]^2\|\le G_A\\ \exists G_B,\ 0\le G_B, & \forall n, & \|\mathbf{B}[n]\|\le G_B\\ \exists G_C,\ 0\le G_C, & \forall n, & \|\mathbf{C}[n]\|\le G_C\\ \exists G_D,\ 0\le G_D, & \forall n, & \|\mathbf{D}[n]\|\le G_D\end{array}\right\}\implies \begin{array}{l}\text{BIBO}\\ \text{Stable}\end{array}.$$

This theorem is hinted at but not proven in [6], and is a special case of the main result of this paper, shown later on in Theorem (2.4). The purpose of this paper is to prove the validity of and expand upon this variant of Criterion 1(1).



(a) Length-2. (b) Length-3.



(c) length-N.

Figure 1: *Delay lines of various lengths:* 2, 3, *and* N.

3. LIMITATIONS OF LAROCHE'S METHOD

3.1. Limitation of Criterion 1(1)

Although Laroche gives some examples of where Criterion 1(1) can be used to prove filter stability, there exists an infinitely large class of time-varying filters which are BIBO stable but cannot be proved BIBO stable using Criterion 1(1). Here we study the simplest such filter: an FIR delay line of length two, as shown in FIg. 1a, which has

$$\left\{\begin{array}{l}\mathbf{A}[n] = \begin{bmatrix}0&0\\1&0\end{bmatrix},\ \mathbf{B}[n] = \begin{bmatrix}1\\0\end{bmatrix}\\ \mathbf{C}[n] = \begin{bmatrix}0&1\end{bmatrix},\ \mathbf{D}[n] = \begin{bmatrix}0\end{bmatrix}.\end{array}\right. \quad (7)$$

This filter is obviously stable according to many criteria:

a) Conventional wisdom: it is an FIR filter with bounded coefficients, with impulse response $h[n] = 0, 0, 1, 0, 0, \ldots!$

b) It is LTI and its eigenvalues have zero magnitude:

$$\left|\begin{bmatrix}0&0\\1&0\end{bmatrix} - \lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}\right| = \left|\begin{bmatrix}-\lambda&0\\1&-\lambda\end{bmatrix}\right| = \lambda^2 = 0 \longrightarrow \begin{array}{c}\lambda=0,\\ |\lambda|<1\end{array}.$$

c) It satisfies the definition of BIBO stability since

$$\forall n,\ \textstyle\sum_{i=-\infty}^{\infty}|h[n,i]| = 1 < 1+\epsilon$$

where ϵ is any finite positive number.

d) The transfer function $H^{\text{FIR-2}}(z) = z^{-2}/1$ has no poles.

e) The filter is allpass and energy-preserving:

$$\sqrt{\textstyle\sum_{n=-\infty}^{\infty}|x[n]|^2} = \sqrt{\textstyle\sum_{n=-\infty}^{\infty}|y[n]|^2}.$$

However, we have $\|\mathbf{A}[n]\| = 1$, because the unit-length vector $\mathbf{x}[n] = \begin{bmatrix}1&0\end{bmatrix}^{\mathrm{T}}$ gets an amplification of 1, i.e., $\|\mathbf{A}[n]\mathbf{x}[n]\| = \|\mathbf{x}[n]\|$. Equivalently, we can say that for

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix} = \begin{bmatrix}1&0\\0&0\end{bmatrix} = \begin{bmatrix}\sigma_1&0\\0&\sigma_2\end{bmatrix} = \begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}. \quad (8)$$

Because $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is diagonal, its eigenvalues λ (and hence the singular values σ of **A**) are its diagonal elements 1 and 0. The largest eigenvalue is 1, hence $\|\mathbf{A}\| = 1$ and Criterion 1(1) fails.

More generally, Laroche's Criterion 1(1) fails for (obviously stable) FIR delay lines of any length, or any filter, including IIR filters, involving delay lines longer than one sample. It also fails for many IIR filters that do not have a delay line longer than one sample, including all power-normalized ladder filters with order greater than 1, which are known to be stable even under time-varying coefficients [4, 5]. An example of this is the 2nd-order power-normalized ladder filter studied in [6]. In [6], when Criterion 1(1) fails, recourse is made to Criterion 2 or Criterion 1(2, frozen).

3.2. Limitation of Criterion 1(2, frozen)

Can we use Criterion 1(2, frozen) (Theorem (2.4)) to prove stability in this case? The square of the state transition matrix is

$$\mathbf{A}[n]^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{9}$$

and the positive semidefinite matrix derived from that product is

$$(\mathbf{A}[n]^2)^{\mathrm{T}} \mathbf{A}[n]^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \tag{10}$$

which has 0 eigenvalues, hence all singular values of $\mathbf{A}[n]^2$ are $0 < 1$, $\left\|\mathbf{A}[n]^2\right\| = 0 < 1$, and BIBO stability is proven.

So, is using this extension always the key to proving stability when $\|\mathbf{A}[n]\| = 1$? Unfortunately, it is not.

Consider the length-3 delay line shown in Fig. 1b which has

$$\begin{cases} \mathbf{A}[n] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, & \mathbf{B}[n] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{C}[n] = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, & \mathbf{D}[n] = \begin{bmatrix} 0 \end{bmatrix}. \end{cases} \tag{11}$$

As with the length-2 delay line, it is obviously stable. We have the following bounds on the system matrices: $\|\mathbf{B}\| = 1$, $\|\mathbf{C}\| = 1$, $\|\mathbf{D}\| = 0$. However, Criterion 1 fails because

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}, \tag{12}$$

which has max eigenvalue 1, hence $\|\mathbf{A}[n]\| = 1$ and Criterion 1(1) fails. Considering Criterion 1(2, frozen), we have

$$\mathbf{M} = \mathbf{A}[n]^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{13}$$

and hence

$$\mathbf{M}^{\mathrm{T}}\mathbf{M} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}, \tag{14}$$

which again has max eigenvalue 1, hence $\left\|\mathbf{A}[n]^2\right\| = 1$ and Criterion 1(2, frozen) fails. This demonstrates that Criterion 1(2, frozen) on its own does not unlock stability proofs for a particularly wide class of filters. We will return to our length-3 delay line and an arbitrary-length delay line in our case studies.

4. MAIN RESULTS: MULTIPLE TIME STEP EXTENSION

Here we present *Criterion 1(μ)*, a variation of Criterion 1(1) applied over μ time steps. The extension that Laroche used in his case study is Criterion 1(2), although we will see in due time that larger μ is possible and necessary for various filters. Our extension bears some similarity to the proof of BIBO stability of time-varying normalized ladder filters given by Phoong and Vaidyanathan [5]. However, our proposed theorem can be used to analyze digital filters with *any* topology or realization, where their work only considers power-normalized ladder filters.

Theorem 4.1 (Criterion 1(μ))**.** *Bounding the product of state transition matrices* $\left\|\prod_{\eta=1}^{\mu} \mathbf{A}[n+\eta-1]\right\| < G_\mu < 1$ *along with any finite bounds on* $\|\mathbf{A}[n]\|$, $\|\mathbf{B}[n]\|$, $\|\mathbf{C}[n]\|$, $\|\mathbf{D}[n]\|$ *for all* n *is a sufficient proof of BIBO stability, for filters whose coefficients may change every sample.*

$$\left.\begin{array}{llr} \exists G_\mu,\ 0 \leq G_\mu < 1, & \forall n, & \left\|\displaystyle\prod_{\eta=1}^{\mu} \mathbf{A}[n+\eta-1]\right\| \leq G_\mu \\ \exists G_A,\ 0 \leq G_A, & \forall n, & \|\mathbf{A}[n]\| \leq G_A \\ \exists G_B,\ 0 \leq G_B, & \forall n, & \|\mathbf{B}[n]\| \leq G_B \\ \exists G_C,\ 0 \leq G_C, & \forall n, & \|\mathbf{C}[n]\| \leq G_C \\ \exists G_D,\ 0 \leq G_D, & \forall n, & \|\mathbf{D}[n]\| \leq G_D \end{array}\right\} \implies \begin{array}{c}\text{BIBO} \\ \text{Stable}\end{array}.$$

Proof. Grouping terms in the product in Lemma (2.1) gives

$$\prod_{\ell=i+1}^{n-1} \mathbf{A}[\ell] = \underbrace{\prod_{\iota=1}^{n-i-1-\mu N_i} \mathbf{A}[n-i-\iota]}_{\|\cdots\|<G_E} \prod_{\beta=0}^{N_i-1} \underbrace{\prod_{\eta=0}^{\mu-1} \mathbf{A}[\mu\beta+i+\eta]}_{\|\cdots\|<G_\mu<1} \tag{15}$$

where $N_i = \left\lfloor \frac{n-i-1}{\mu} \right\rfloor$ and $\lfloor\cdot\rfloor$ is the floor function. Now, via submultiplicativity of the matrix norm, we can write

$$\left\|\prod_{\ell=i+1}^{n-1} \mathbf{A}[\ell]\right\| \leq G_E G_\mu^{N_i}, \tag{16}$$

where each G_μ comes from one of the N_i right braced product terms and G_E represents the finite upper bound of the left braced product terms, which must exist since they are a product of at most $\mu - 1$ terms, each of which is bounded by G_A. The matrix norm is submultiplicative, so we can return to Lemma (2.1) and, using (16) and the bounds on $\mathbf{B}[n]$, $\mathbf{C}[n]$, $\mathbf{D}[n]$, write

$$h_\infty[n] \leq G_D + \sum_{i=-\infty}^{n-1} G_C G_E G_\mu^{N_i} G_B. \tag{17}$$

The term $\sum_{i=-\infty}^{n-1} G_C G_E G_\mu^{N_i} G_B$ in (17) can be rewritten as

$$G_C G_E \left(\sum_{i=-\infty}^{n-1} G_\mu^{N_i}\right) G_B = G_C G_E \left(\sum_{\iota=0}^{\infty} G_\mu^{\left\lfloor \frac{\iota}{\mu} \right\rfloor}\right) G_B. \tag{18}$$

The infinite sum term is the sum of μ geometric series.

$$\sum_{\iota=0}^{\infty} G_\mu^{\left\lfloor \frac{\iota}{\mu} \right\rfloor} = \mu \sum_{\iota=0}^{\infty} G_\mu^{\iota} = \frac{\mu}{1-G_\mu}, \tag{19}$$

where each infinite series converge because $G_\mu < 1$.

Now, since $G_D + \mu\frac{G_B G_C G_E}{1-G_\mu}$ is finite, it can be bounded by G, so we can rewrite (17) as

$$\exists G,\ \forall n, \quad h_\infty[n] = G_D + \mu\frac{G_B G_C G_E}{1-G_\mu} < G, \tag{20}$$

proving BIBO stability. ■

Corollary 4.1.1. *Criterion 1(1) is proven as a special cases of Theorem* (4.1)*, where* $\mu = 1$.

Corollary 4.1.2. *Criterion 1(2, frozen) is proven as a special cases of Theorem* (4.1)*, where* $\mu = 2$ *and "frozen" coefficients are considered a special case of freely varying coefficients.*

5. A HELPFUL THEOREM

With large matrices, it can be very difficult or inconvenient to find the eigenvalues of $\mathbf{M}^\mathrm{T}\mathbf{M}$. Here we present a Lemma that can be used to avoid this hassle. This Lemma allows us to prove that all eigenvalues of $\mathbf{M}^\mathrm{T}\mathbf{M}$ are less than 1 by looking only at the determinant det $\left[\mathbf{M}^\mathrm{T}\mathbf{M}\right]$ and trace tr $\left[\mathbf{M}^\mathrm{T}\mathbf{M}\right]$. The trace especially is simple to calculate: it is the sum of the diagonal entries of a matrix.

We define a sum Σ and product Π of the N singular values $\sigma_1, \sigma_2, \cdots, \sigma_N \in \mathbb{R}_+$ of a square matrix $\mathbf{M} \in \mathbb{R}^{N\times N}$

$$\sigma_1 + \sigma_2 + \cdots + \sigma_N = \Sigma, \qquad \sigma_1\sigma_2\cdots\sigma_N = \Pi, \tag{21}$$

These relate to the trace and determinant [8] of $\mathbf{M}^\mathrm{T}\mathbf{M}$ by

$$\Sigma = \mathrm{tr}\left[\mathbf{M}^\mathrm{T}\mathbf{M}\right], \qquad \Pi = \det\left[\mathbf{M}^\mathrm{T}\mathbf{M}\right]. \tag{22}$$

Lemma 5.1. *Given N real, non-negative numbers, if their product is less than* 1 *and their sum is less that* 1 *plus their product, then all of the numbers are* < 1.

$$\left.\begin{matrix}\Pi < 1\\ \Sigma < 1 + \Pi\end{matrix}\right\} \Longrightarrow \sigma_k < 1,\ k \in \{1, 2, \cdots, N\}.$$

Proof. Assume without loss of generality that $\sigma_1 \le \sigma_2 \cdots \le \sigma_N$. If $\sigma_N < 1$, then all $\sigma_k < 1$ and the conclusion holds.

However, assuming $1 \le \sigma_N$ leads to a contradiction. By our hypothesis, $\sum_{k\le N-1}\sigma_k + \sigma_N < 1 + \prod_{k\le N}\sigma_k$. Using our hypothesis that $\Pi < 1$, this implies $\sum_{k\le N-1}\sigma_k + \sigma_N < 2$. Using our assumption that $1 \le \sigma_N$, we have $\sum_{k\le N-1}\sigma_k < 1$, which implies $\sigma_k < 1$ for all $k \le N-1$ since each summand must be less than or equal to the sum when all summands are non-negative. From this we see $\prod_{k\le N-1}\sigma_k \le \sigma_j$ for all $j \le N-1$, which holds for non-negative numbers less than 1. Thus, $\prod_{k\le N-1}\sigma_k \le \sum_{k\le N-1}\sigma_k$, again since each summand must be less than or equal to the sum. Combining this with our hypothesis that $\Sigma < 1+\Pi$, we see $\sum_{k\le N-1}\sigma_k + \sigma_N < 1 + \sigma_N\sum_{k\le n-1}\sigma_k$. Rearranging, we have $0 < (\sigma_N - 1)(\sum_{k\le n-1}\sigma_k - 1)$. By our assumption $1 \le \sigma_N$, we know the first term is non-negative, which implies the second term is non-negative. But this contradicts the fact that $\sum_{k\le N-1}\sigma_k < 1$, so we must have $\sigma_N < 1$. ■

Theorem 5.2. *Given a square matrix* $\mathbf{M} \in \mathbb{R}^{N\times N}$ *representing the product of many state transition matrices over time, if det*$(\mathbf{M}) < 1$ *and tr*$(\mathbf{M}) < 1 +$ *det*$(\mathbf{M})$*, then the time-varying system represented by* $\mathbf{M}$ *is BIBO stable.*

Proof. By Lemma (5.1), all of the eigenvalues of $\mathbf{M}^\mathrm{T}\mathbf{M}$ are < 1. Therefore, by Lemma (2.3), all of the singular values of $\mathbf{M}$ are < 1 and $\|\mathbf{M}\| < 1$. By Theorem (4.1), this means that the time-varying filter is BIBO stable. ■

6. CASE STUDIES

6.1. Delay Lines

6.1.1. Length-3 Delay Line

First we return to the length-3 delay line, which could not be proven stable using either Criterion 1(1) or Criterion 1(2, frozen). Here we show that using Criterion 1(μ), $3 \le \mu$, we can now prove it stable. We also emphasize that we will do no "freezing" of the filter coefficicents, although that hardly matters for this particular filter. Considering μ time steps, i.e., $\mathbf{M} = \mathbf{A}[n]\mathbf{A}[n-1]\cdots\mathbf{A}[n-\mu+1]$, we have

$$\mathbf{M} = \begin{cases}\begin{bmatrix}\mathbf{0}_{\mu\times(3-\mu)} & \mathbf{0}_\mu\\ \mathbf{I}_{3-\mu} & \mathbf{0}_{(3-\mu)\times\mu}\end{bmatrix}, & \mu < 3\\ \begin{bmatrix}\mathbf{0}_{3\times3}\end{bmatrix}, & 3 \le \mu\end{cases} \tag{23}$$

$$\mathbf{M}^\mathrm{T}\mathbf{M} = \begin{cases}\begin{bmatrix}\mathbf{I}_{3-\mu} & \mathbf{0}_{(3-\mu)\times\mu}\\ \mathbf{0}_{\mu\times(3-\mu)} & \mathbf{0}_\mu\end{bmatrix}, & \mu < 3\\ \begin{bmatrix}\mathbf{0}_3\end{bmatrix}, & 3 \le \mu\end{cases} \tag{24}$$

where $\mathbf{0}_{a\times b}$ is the $a \times b$ zero matrix, $\mathbf{0}_a$ is the square, $a \times a$ zero matrix, and $\mathbf{I}_a$ is the square $a \times a$ identity matrix. From this, we can see that the eigenvalues of $\mathbf{M}^\mathrm{T}\mathbf{M}$ are 1 with multiplicity $\max(3-\mu, 0)$ and 0 with multiplicity $\min(\mu, 3)$. Therefore, we need $3 \le \mu$ to prove BIBO stability using Criterion 1(μ). To reduce the effort, the lowest value of $\mu = 3$ should be used.

6.1.2. Length-N Delay Line

If instead we consider a length-N delay line, shown in Fig. 1c, then our system matrices are

$$\begin{cases}\mathbf{A} = \begin{bmatrix}\mathbf{0}_{1\times(N-1)} & 0\\ \mathbf{I}_{N-1} & \mathbf{0}_{(N-1)\times1}\end{bmatrix}, & \mathbf{B} = \begin{bmatrix}1\\ \mathbf{0}_{(N-1)\times1}\end{bmatrix}\\ \mathbf{C} = \begin{bmatrix}\mathbf{0}_{(N-1)\times1} & 1\end{bmatrix}, & \mathbf{D} = \begin{bmatrix}0\end{bmatrix}.\end{cases} \tag{25}$$

As with the length-2 delay line, $\mathbf{A}[n]$ for $N \ge 2$ fails Criterion 1(1). Considering $\mathbf{M} = \mathbf{A}[n]\mathbf{A}[n-1]$, we have

$$\mathbf{M} = \begin{bmatrix}\mathbf{0}_{2\times(N-2)} & \mathbf{0}_2\\ \mathbf{I}_{N-2} & \mathbf{0}_{(N-2)\times2}\end{bmatrix}, \tag{26}$$

$$\mathbf{M}^\mathrm{T}\mathbf{M} = \begin{bmatrix}\mathbf{I}_{N-2} & \mathbf{0}_{(N-2)\times2}\\ \mathbf{0}_{2\times(N-2)} & \mathbf{0}_2\end{bmatrix}. \tag{27}$$

where the eigenvalues of $\mathbf{M}^\mathrm{T}\mathbf{M}$ (and hence the singular values of $\mathbf{A}[n]\mathbf{A}[n-1]$) are 0 (multiplicity of 2) and 1 (multiplicity of $N-2$), and hence whose matrix norm is $\|\mathbf{A}[n]\mathbf{A}[n-1]\| = 1$. So, BIBO stability cannot be proven with Criterion 1(2).

Now we will consider the product over μ timesteps, i.e., $\mathbf{M} = \mathbf{A}[n]\mathbf{A}[n-1]\cdots\mathbf{A}[n-\mu+1]$. This gives us

$$\mathbf{M} = \begin{cases}\begin{bmatrix}\mathbf{0}_{\mu\times(N-\mu)} & \mathbf{0}_\mu\\ \mathbf{I}_{N-\mu} & \mathbf{0}_{(N-\mu)\times\mu}\end{bmatrix}, & \mu < N\\ \begin{bmatrix}\mathbf{0}_N\end{bmatrix}, & N \le \mu\end{cases} \tag{28}$$

$$\mathbf{M}^\mathrm{T}\mathbf{M} = \begin{cases}\begin{bmatrix}\mathbf{I}_{N-\mu} & \mathbf{0}_{(N-\mu)\times\mu}\\ \mathbf{0}_{\mu\times(N-\mu)} & \mathbf{0}_\mu\end{bmatrix}, & \mu < N\\ \begin{bmatrix}\mathbf{0}_N\end{bmatrix}, & N \le \mu\end{cases} \tag{29}$$

This shows us that the singular values of $\mathbf{M}$ are 0 (with multiplicity μ for $\mu < N$ and N otherwise) and 1 (with multiplicity $N-\mu$ for $\mu < N$ and 0 otherwise). So, we have

$$\|\mathbf{M}\| = \begin{cases}1, & \mu < N\\ 0, & N \le \mu\end{cases}. \tag{30}$$

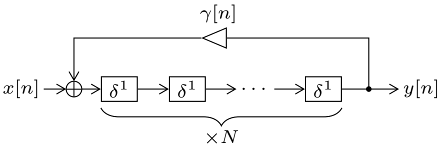


Figure 2: *A simple feedback delay network (FDN) with a length-N delay line and feedback gain $\gamma[n]$.*

So, Criterion $1(N-1)$ and below will fail, whereas using Criterion $1(N)$ and above can be used to show BIBO stability.

For the length-N delay line, this result can be explained qualitatively. An impulse in any state will eventually pass through the delay line. The impulse that takes the longest to disappear is the one entering the beginning of the delay line. Watching how it travels down the line and affects the norm of the state vector, we see:

n	$\mathbf{x}[n+1] = \left(\prod_{\eta=1}^{n} \mathbf{A}[\eta]\right) \mathbf{x}[0]$	$\lVert\mathbf{x}[n]\rVert$
0	$\begin{bmatrix}1 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}^{\mathrm{T}}$	1
1	$\begin{bmatrix}0 & 1 & 0 & \cdots & 0 & 0\end{bmatrix}^{\mathrm{T}}$	1
$\vdots$	$\vdots$	$\vdots$
$N-1$	$\begin{bmatrix}0 & 0 & 0 & \cdots & 0 & 1\end{bmatrix}^{\mathrm{T}}$	1
N	$\underbrace{\begin{bmatrix}0 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}^{\mathrm{T}}}_{N}$	0

The length of $\mathbf{x}[n]$ remains at 1 until step N, when it drops to 0. This can give us some intuition about the number of time steps that need to be considered in a proof for a particular filter.

6.2. One-Channel Feedback Delay Networks (FDNs)

Now we consider an IIR filter, a length-N delay line that feeds back into itself with a gain $\gamma[n]$. This simple Feedback Delay Network (FDN) [9] is shown in Fig. 2. Its system matrices are

$$\begin{cases} \mathbf{A} = \begin{bmatrix} \mathbf{0}_{1\times(N-1)} & \gamma[n] \\ \mathbf{I}_{N-1} & \mathbf{0}_{(N-1)\times 1} \end{bmatrix}, & \mathbf{B} = \begin{bmatrix} 1 \\ \mathbf{0}_{(N-1)\times 1} \end{bmatrix} \\ \mathbf{C} = \begin{bmatrix} \mathbf{0}_{(N-1)\times 1} & 1 \end{bmatrix}, & \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}. \end{cases} \tag{31}$$

Now we will consider the product over μ timesteps, i.e., $\mathbf{M} = \mathbf{A}[n]\mathbf{A}[n-1]\cdots\mathbf{A}[n-\mu+1]$. This gives us

$$\mathbf{M} = \begin{bmatrix} \mathbf{0}_{1\times(N-1)} & 1 \\ \mathbf{I}_{N-1} & \mathbf{0}_{(N-1)\times 1} \end{bmatrix}^{\mu} \begin{bmatrix} \Gamma_{\mu,N-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Gamma_{\mu,0} \end{bmatrix} \tag{32}$$

where

$$\Gamma_{\mu,\rho} = \prod_{\eta=0}^{\lceil(\mu-\rho-1)/N\rceil} \gamma[n-\rho-\eta N] \tag{33}$$

and $\lceil\cdot\rceil$ is the ceiling function. We then have

$$\mathbf{M}^{\mathrm{T}}\mathbf{M} = \begin{bmatrix} \Gamma^2_{\mu,N-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Gamma^2_{\mu,0} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_N \end{bmatrix} \tag{34}$$

from which we can see the eigenvalues are $\Gamma^2_{\mu,0}, \cdots, \Gamma^2_{\mu,N-1}$. So, it is clear from the definition of $\Gamma_{\mu,\rho}$ that

$$\lVert\mathbf{M}\rVert = \max_{\nu\in[0,N-1]} |\Gamma_{\mu,\nu}| \tag{35}$$

and that we will get $\lVert\mathbf{M}\rVert < 1$ when $N \leq \mu$, so long as $|\gamma[n]| < 1$, $\forall n$. We emphasize that there is no condition that the filter coefficients be "frozen" for any span of time.

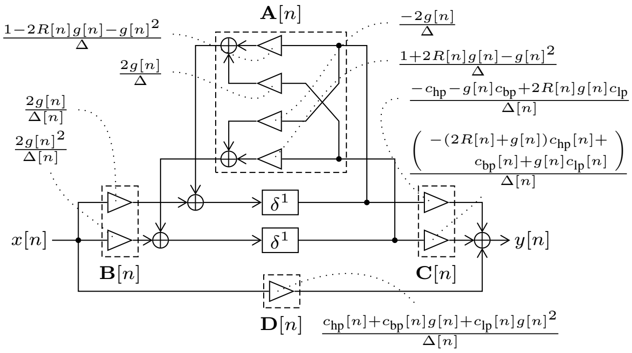


Figure 3: *A two-pole State-Variable Filter (SVF): an IIR filter which cannot be proven stable with Laroche's Criterion 1(1), which can be proven stable with Criterion 1(2).*

We have the following bounds on the other system matrices: $\lVert\mathbf{A}\rVert = 1$, $\lVert\mathbf{B}\rVert = 1$, $\lVert\mathbf{C}\rVert = 1$, $\lVert\mathbf{D}\rVert = 0$. So, with those and the bound on $\lVert\mathbf{M}\rVert$, we can prove BIBO stability using Criterion $1(\mu)$.

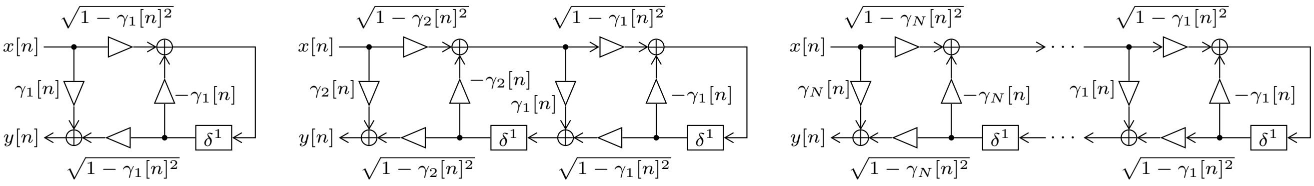
Similar to the length-N delay line, this result can be explained qualitatively. An impulse in any state will eventually circulate around the delay line and be contracted by γ. The impulse that takes the longest to get contracted is the one entering the beginning of the delay line. Watching how it travels down the line and affects the norm of the state vector, we see:

n	$\mathbf{x}[n+1] = \left(\prod_{\eta=1}^{n} \mathbf{A}[\eta]\right) \mathbf{x}[0]$	$\lVert\mathbf{x}[n]\rVert$
0	$\begin{bmatrix}1 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}^{\mathrm{T}}$	1
1	$\begin{bmatrix}0 & 1 & 0 & \cdots & 0 & 0\end{bmatrix}^{\mathrm{T}}$	1
$\vdots$	$\vdots$	$\vdots$
$N-1$	$\begin{bmatrix}0 & 0 & 0 & \cdots & 0 & 1\end{bmatrix}^{\mathrm{T}}$	1
N	$\underbrace{\begin{bmatrix}\gamma[N-1] & 0 & 0 & \cdots & 0 & 0\end{bmatrix}^{\mathrm{T}}}_{N}$	$\lvert\gamma[N-1]\rvert$

On a more complex level, this idea was used in [3] to conjecture that the number of timesteps needed for a unitary FDN is the length of the longest delay line. To put it more generally and precisely, in any time-varying digital filter, the lower bound on number of time steps needed is length of longest delay line.

6.3. Two-pole State Variable Filter (SVF)

An important time-varying filter for musical applications is the two-pole State Variable Filter (SVF) discretized using Trapezoidal Transposed Direct Form II integrators, as described in [10] and shown in Fig. 3. This structure conveniently implements common filter types such as low-pass and high-pass filters, and retains stability even when coefficients are varied each sample. The filter is parameterized by five potentially time-varying coefficients: $\gamma[n]$, $R[n]$, $c_{\text{hp}}[n]$, $c_{\text{bp}}[n]$, and $c_{\text{lp}}[n]$. Only $\gamma[n]$, $R[n]$ appear in $\mathbf{A}$ so as long as $c_{\text{hp}}[n]$, $c_{\text{bp}}[n]$, and $c_{\text{lp}}[n]$ are bounded they cannot contribute to instability. Its state-space description is



(a) 1st order. (b) 2nd order. (c) Nth order.

Figure 4: *Various power-normalized ladder filters, of order* 1, 2, *and* N, *in allpass output configuration.*

$$\begin{cases} \mathbf{A}[n] = \begin{bmatrix} \frac{1-2Rg-g^2}{\Delta} & \frac{-2g}{\Delta} \\ \frac{2g}{\Delta} & \frac{1+2Rg-g^2}{\Delta} \end{bmatrix}, \ \mathbf{B}[n] = \begin{bmatrix} \frac{2g}{\Delta} \\ \frac{2g^2}{\Delta} \end{bmatrix} \\ \mathbf{C}[n] = \begin{bmatrix} \frac{-c_{\text{hp}}(2R+g)+c_{\text{bp}}+c_{\text{lp}}g}{\Delta} & \frac{-c_{\text{hp}}-c_{\text{bp}}g+c_{\text{lp}}(2Rg+1)}{\Delta} \end{bmatrix} \\ \mathbf{D}[n] = \begin{bmatrix} \frac{c_{\text{hp}}+c_{\text{bp}}g+c_{\text{lp}}g^2}{\Delta} \end{bmatrix}. \end{cases} \tag{36}$$

where $\Delta = 1 + 2R[n]g[n] + g[n]^2$. The time index $[n]$ on all quantities has been suppressed for compactness. [10] showed this filter to be stable under time-varying γ and R, under the condition

$$\gamma[n] > 0, R[n] > 0, \forall n. \tag{37}$$

Here we provide a different proof of stability using Theorem (5.2). We use two time steps to define $\mathbf{M} = \mathbf{A}[n]\mathbf{A}[n-1]$. To satisfy the hypotheses of (5.2), we must show that $\Pi < 1$ and $\Sigma < 1 + \Pi$ where Π and Σ represent respectively the determinant and trace of $\mathbf{M}^{\mathrm{T}}\mathbf{M} = \mathbf{A}[n-1]^{\mathrm{T}}\mathbf{A}[n]^{\mathrm{T}}\mathbf{A}[n]\mathbf{A}[n-1]$. Through algebraic manipulation, we can see that $\Sigma < 1 + \Pi$ is true when

$$64R_{[n-1]}R_{[n]}\gamma_{[n-1]}\gamma_{[n]}\left(\gamma_{[n-1]}^2 + 2\gamma_{[n-1]}\gamma_{[n]} + \gamma_{[n]}^2\right) > 0 \tag{38}$$

and $\Pi < 1$ is true when

$$\begin{aligned} &32R_{[n-1]}^2R_{[n]}\gamma_{[n-1]}^2\gamma_{[n]}^3 + 32R_{[n-1]}^2R_{[n]}\gamma_{[n-1]}^2\gamma_{[n]} \\ &\quad + 32R_{[n-1]}R_{[n]}^2\gamma_{[n-1]}^3\gamma_{[n]}^2 + 32R_{[n-1]}R_{[n]}^2\gamma_{[n-1]}\gamma_{[n]}^2 \\ &\quad + 8R_{[n-1]}\gamma_{[n-1]}^3\gamma_{[n]}^4 + 16R_{[n-1]}\gamma_{[n-1]}^3\gamma_{[n]}^2 \\ &\quad + 8R_{[n-1]}\gamma_{[n-1]}^3 + 8R_{[n-1]}\gamma_{[n-1]}\gamma_{[n]}^4 \\ &\quad + 16R_{[n-1]}\gamma_{[n-1]}\gamma_{[n]}^2 + 8R_{[n-1]}\gamma_{[n-1]} + 8R_{[n]}\gamma_{[n-1]}^4\gamma_{[n]}^3 \\ &\quad + 8R_{[n]}\gamma_{[n-1]}^4\gamma_{[n]} + 16R_{[n]}\gamma_{[n-1]}^2\gamma_{[n]}^3 + 16R_{[n]}\gamma_{[n-1]}^2\gamma_{[n]} \\ &\quad + 8R_{[n]}\gamma_{[n]}^3 + 8R_{[n]}\gamma_{[n]} > 0 \end{aligned} \tag{39}$$

Both (38) and (39) are always satisfied under the condition (37) since both are inequalities on polynomial expressions of $\gamma[n], \gamma[n-1], R[n], R[n-1]$ with positive coefficients on each term. This result was checked using the SymPy [11] computer algebra system.

6.4. Power-normalized ladder

Here we consider power-normalized ladder filters of order 1, 2, and $0 < N$. This type of filter has been known to be L2 stable with time-varying coefficients since its introduction in [4], and its BIBO stability with time-varying coefficients was first proved in [5]. Stability of these structures can also be proven via a physical analogy, as in digital waveguide modeling [12]. We will demonstrate that Laroche's techniques fail to prove BIBO stability for ladders with more than two stages whereas our proposed Theorem can prove stability for larger structures. We consider the allpass output for simplicity, but the proofs would be very similar for other ladder filters without zero "tap" coefficients.

6.4.1. *1st-order*

For 1st-order, shown in Fig. 4a, the state-space description is

$$\begin{cases} \mathbf{A}[n] = \begin{bmatrix} -\gamma[n] \end{bmatrix}, \ \mathbf{B}[n] = \begin{bmatrix} \kappa[n] \end{bmatrix} \\ \mathbf{C}[n] = \begin{bmatrix} \kappa[n] \end{bmatrix}, \ \mathbf{D}[n] = \begin{bmatrix} \gamma[n] \end{bmatrix}. \end{cases} \tag{40}$$

where $\kappa[n] = \sqrt{1-\gamma[n]^2}$. $\gamma[n]$ is bounded to $-1 < \gamma[n] < 1$, $\forall n$. So, we have the bounds $\|\mathbf{B}[n]\|, \|\mathbf{C}[n]\|, \|\mathbf{D}[n]\| < 1$, $\forall n$. So it only remains to study the matrix norm of $\mathbf{A}[n]$. We can immediately calculate

$$\mathbf{M}^{\mathrm{T}}\mathbf{M} = \mathbf{A}[n]^{\mathrm{T}}\mathbf{A}[n] = \begin{bmatrix} \gamma[n]^2 \end{bmatrix} \tag{41}$$

which yields

$$\Sigma = \mathrm{tr}[\mathbf{M}^{\mathrm{T}}\mathbf{M}] = \gamma[n]^2, \qquad \Pi = \det[\mathbf{M}^{\mathrm{T}}\mathbf{M}] = \gamma[n]^2. \tag{42}$$

These obviously satisfy $\Pi < 1$ and $\Sigma < 1 + \Pi$, therefore this structure is BIBO stable according to Laroche's Criterion 1.

6.4.2. *2nd-order*

For 2nd-order, shown in Fig. 4b, the state-space description is

$$\begin{cases} \mathbf{A} = \begin{bmatrix} -\gamma_1[n] & -\gamma_2[n]\kappa_1[n] \\ \kappa_1[n] & -\gamma_1[n]\gamma_2[n] \end{bmatrix}, & \mathbf{B} = \begin{bmatrix} \kappa_1[n]\kappa_2[n] \\ \kappa_2[n]\gamma_1[n] \end{bmatrix} \\ \mathbf{C} = \begin{bmatrix} 0 & \kappa_2[n] \end{bmatrix}, & \mathbf{D} = \begin{bmatrix} \gamma_2[n] \end{bmatrix}. \end{cases} \tag{43}$$

where $\kappa_i[n] = \sqrt{1-\gamma_i[n]^2}$. We have $-1 < \gamma_i[n], \kappa_i[n] < +1, \forall i, n$. This leads to the bounds $\|\mathbf{B}[n]\|, \|\mathbf{C}[n]\|, \|\mathbf{D}[n]\| < 1, \forall n$. However $\|\mathbf{A}[n]\| = 1$, so Criterion 1(1) cannot be used directly.

Laroche studied $\mathbf{A}[n]^2 = \mathbf{A}[n]\mathbf{A}[n]$, showing $\left\|\mathbf{A}[n]^2\right\| < 1$, and claiming that this means a 2nd-order ladder filter with coefficients frozen over two time steps is guaranteed stable [6]. We can show, for this filter, that we do not need to restrict the coefficients to be frozen across the two time steps. We will study

$$\mathbf{M} = \mathbf{A}[n]\mathbf{A}[n-1] = \begin{bmatrix} -\gamma_1 & -\gamma_2\kappa_1 \\ \kappa_1 & -\gamma_1\gamma_2 \end{bmatrix} \begin{bmatrix} -\gamma_1' & -\gamma_2'\kappa_1' \\ \kappa_1' & -\gamma_1'\gamma_2' \end{bmatrix} \tag{44}$$

$$= \begin{bmatrix} \gamma_1\gamma_1' + \gamma_2\kappa_1\kappa_1' & \gamma_1\gamma_2'\kappa_1' + \gamma_2\kappa_1\gamma_1'\gamma_2' \\ -\kappa_1\gamma_1' - \gamma_1\gamma_2\kappa_1' & -\kappa_1\gamma_2'\kappa_1' + \gamma_1\gamma_2\gamma_1'\gamma_2' \end{bmatrix}, \tag{45}$$

where no time index compactly indicates $[n]$ and $'$ indicates $[n-1]$.

$$\mathbf{V} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] = \left[\begin{array}{cccccc|c} -\gamma_1 & -\kappa_1\gamma_2 & -\kappa_1\kappa_2\gamma_3 & \cdots & -\kappa_1\cdots\kappa_{N-2}\gamma_{N-1} & -\kappa_1\cdots\kappa_{N-1}\gamma_N & \kappa_1\cdots\kappa_N \\ \kappa_1 & -\gamma_1\gamma_2 & -\gamma_1\kappa_2\gamma_3 & \cdots & -\gamma_1\kappa_2\cdots\kappa_{N-2}\gamma_{N-1} & -\gamma_1\kappa_2\cdots\kappa_{N-1}\gamma_N & \gamma_1\kappa_2\cdots\kappa_N \\ 0 & \kappa_2 & -\gamma_2\gamma_3 & \cdots & -\gamma_2\kappa_3\cdots\kappa_{N-2}\gamma_{N-1} & -\gamma_2\kappa_3\cdots\kappa_{N-1}\gamma_N & \gamma_2\kappa_3\cdots\kappa_N \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\gamma_{N-2}\gamma_{N-1} & -\gamma_{N-2}\kappa_{N-1}\gamma_N & \gamma_{N-2}\kappa_{N-1}\kappa_N \\ 0 & 0 & 0 & \cdots & \kappa_{N-1} & \gamma_{N-1}\gamma_N & -\gamma_{N-1}\kappa_N \\ \hline 0 & 0 & 0 & \cdots & 0 & \kappa_N & \gamma_N \end{array}\right]$$

Figure 5: *System matrix for Nth-order power-normalized ladder filter, where every term's time index $[n]$ is suppressed for compactness.*

From here we can define Σ and Π to be the trace resp. determinant of $\mathbf{M}^{\mathrm{T}}\mathbf{M}$. Through algebraic manipulation we find

$$\Sigma = (\gamma_1')^2\left((\gamma_2')^2\gamma_2^2 - (\gamma_2')^2 - \gamma_2^2 + 1\right) + (\gamma_2')^2 + \gamma_2^2 \quad (46)$$

$$\Pi = \gamma_2[n-1]^2\gamma_2[n]^2. \quad (47)$$

To use (5.2) to show this filter is stable, we must show $\Sigma < 1 + \Pi$ and $\Pi < 1$ are true. Through algebraic manipulation we can see that $\Sigma < 1 + \Pi$ is true when

$$\left(\gamma_1' - 1\right)\left(\gamma_1' + 1\right)\left(\gamma_2' - 1\right)\left(\gamma_2' + 1\right)\left(\gamma_2 - 1\right)\left(\gamma_2 + 1\right) < 0. \quad (48)$$

Likewise we can see that $\Pi < 1$ holds when

$$\gamma_2[n-1]^2\gamma_2[n]^2 < 1 \quad (49)$$

Since we have $-1 < \gamma_1[n] < 1, -1 < \gamma_2[n] < 1, \forall n$, we can see that (48) always holds, since the left side is a product of 3 negative and 3 positive terms, yielding a negative number. (49) also always holds, since both terms of the product are positive (< 1).

6.4.3. Nth-order

Here we study an Nth-order power-normalized ladder filter, shown in Fig. 4c, whose system matrix is shown in Fig. 5.

For this particular type of filter, notice that the system matrix $\mathbf{V} \in \mathbb{R}^{(N+1)\times(N+1)}$ can be factored as [13]

$$\mathbf{V}[n] = \mathbf{F}_1[n]\cdots\mathbf{F}_N[n], \quad (50)$$

which lets us find the state matrix as

$$\mathbf{A}[n] = \underbrace{\begin{bmatrix}\mathbf{I}_N & \mathbf{0}_{N\times 1}\end{bmatrix}}_{\text{only term that can contract!}} \mathbf{F}_1[n]\cdots\mathbf{F}_N[n]\begin{bmatrix}\mathbf{I}_N \\ \mathbf{0}_{1\times N}\end{bmatrix} \in \mathbb{R}^{N\times N} \quad (51)$$

where, for $i \in \{1, \cdots, N\}$,

$$\mathbf{F}_i[n] = \begin{bmatrix} \mathbf{I}_{i-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma_i[n] & \kappa_i[n] & \mathbf{0} \\ \mathbf{0} & \kappa_i[n] & -\gamma_i[n] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{N-i} \end{bmatrix} \in \mathbb{U}^{(N+1)\times(N+1)}. \quad (52)$$

Each $\mathbf{F}_i[n]$ is a reflection across angle $\theta_i = \arccos\left(\gamma_i[n]/2\right)$ in the dimensions i and $i+1$. Being reflections, each $\mathbf{F}_i[n]$ is unitary. Because $\mathbf{A}[n]$ is the lower right partition of the product (51), we can see that the product $\mathbf{F}_1[n]\cdots\mathbf{F}_N[n]$ is unitary, so does not affect the length of any vector, and that $\left[\begin{smallmatrix}\mathbf{I}_N \\ \mathbf{0}_{1\times N}\end{smallmatrix}\right]$ cannot contract the norm of any vector. It's *only* the first term $\left[\begin{smallmatrix}\mathbf{I}_N & \mathbf{0}_{N\times 1}\end{smallmatrix}\right]$ that can affect the norm of a vector, since it truncates the last dimension.

The class of vectors that would not be contracted when left-multiplied by $\mathbf{A}[n]$ is all those that are zero in their last dimension,

$$\mathcal{X} = \begin{bmatrix}\dagger & \cdots & \dagger & 0\end{bmatrix}^{\mathrm{T}}, \quad (53)$$

where $\dagger$ represents arbitrary values. Conversely, the complementary class of vectors that will be contracted when left-multiplied by $\mathbf{A}[n]$ is those that are not zero in last dimension, i.e.,

$$\overline{\mathcal{X}} = \begin{bmatrix}\dagger & \cdots & \dagger & \mathbb{R}_{\setminus 0}\end{bmatrix}^{\mathrm{T}}. \quad (54)$$

$\overline{\mathcal{X}} = \varnothing$ for $N = 1$, but is not empty for $1 < N$. For instance, for $1 < N$ we can always consider

$$\mathbf{x}_{\text{outer}}[n] = \begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}^{\mathrm{T}} \in \overline{\mathcal{X}}. \quad (55)$$

which results in the vector

$$\mathbf{x}_{\text{outer}}[n+1] = \mathbf{A}[n]\mathbf{x}_{\text{outer}}[n] \quad (56)$$

which has the property $\|\mathbf{x}_{\text{outer}}[n+1]\| < 1$.

Because there exist state vectors, e.g. $\mathbf{x}_{\text{outer}}$, which are not contracted for $1 < N$, i.e, $\mathcal{X} \neq \varnothing$, that means that $\|\mathbf{M}[n]\| = 1$ for $1 < N$, and stability cannot be proven using Theorem (2.2).

Therefore, for $1 < N$, we cannot use Criterion 1(1), and must use Criterion 1(μ), for some $1 < \mu$. But, how many time steps will we need, i.e., what is the smallest value of μ that will work? For this particular class of filters, we can answer this question.

We want to find some μ such that for any non-zero vector $\mathbf{x}[n]$, $\mathbf{x}[n]$ is contracted by $\mathbf{M} = \prod_{\eta=0}^{\mu-1} A[n+\eta]$. We know that $\mathbf{x}[n]$ has a non-zero value in at least one position. We call the highest index with a non-zero value m. If $m = N$, then $\mathbf{x} \in \overline{\mathcal{X}}$, so $\mathbf{A}$ will act as a contraction. We know for all η, $\|\mathbf{A}[\eta]\| = 1$, so multiplying by further $\mathbf{A}$ terms will not "recover" from this contraction[1] and $\mathbf{x}[n]$ will be contracted by $\mathbf{M}$. On the other hand, if $m < N$, multiplying by $\mathbf{A}[n]$ will always create a non-zero value in position $m+1$, so $\mathbf{x}[n+1]$ will now have $m+1$ as the highest index with a non-zero value. Recalling that the restriction $|\gamma_i[n]| < 1$ means that $0 < \kappa_i[n] \leq 1$, this is due to the structure of the factorization shown in (51). When $\mu = N$, we have N terms in $\mathbf{M} = \prod_{\eta=0}^{N-1} A[n+\eta]$, and each one will either contract $\mathbf{x}$ or increase the highest non-zero index of $\mathbf{x}$ by 1, so as $\mathbf{x}$ passes through these steps, for at least one $\eta < N$, $\mathbf{x}[n+\eta] \in \overline{\mathcal{X}}$, thus $\mathbf{M}$ will contract $\mathbf{x}[n]$, and since $\mathbf{x}[n]$ is arbitrary, we have $\|M\| < 1$.

Similar to the delay line, we can illustrate this by considering the "worst-case" unit impulse, the one entering the "innermost" delay in the structure (on the right side of Fig. 4c.

$$\mathbf{x}_{\text{inner}} = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}^{\mathrm{T}} \in \mathcal{X}. \quad (57)$$

---

[1] More formally, this is due to sub-multiplicativity of the norm:

$$\left\|\prod_{\eta=0}^{\mu} A[n+\eta]\right\| \leq \|A[n+\mu]\| \left\|\prod_{\eta=0}^{\mu-1} A[n+\eta]\right\|.$$

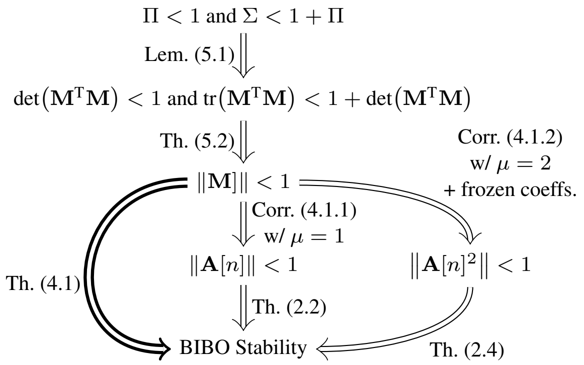


Figure 6: *Implication graph of the theorems used in this paper, with the main theorem bolded.*

At each time step, the index of the highest non-zero value of $\mathbf{x}[n+1] = \left(\prod_{\eta=1}^{n}\mathbf{A}[\eta]\right)\mathbf{x}_{\text{inner}}$ grows. That is, taking $\mathbf{x}[0] = \mathbf{x}_{\text{inner}}$,

$$\mathbf{x}[n] = \begin{bmatrix}[*]_{1\times(n+1)} & \mathbf{0}_{1\times(N-n)}\end{bmatrix}^{\mathrm{T}} \in \mathcal{X}, \quad \text{for } 0 < n < N \tag{58}$$

where $[*]_{1\times n}$ is some vector of unit norm ($\|[*]\| = 1$). We can see that after $N-1$ time steps, we will have

$$\mathbf{x}[N-1] = [*]^{\mathrm{T}}_{1\times N} \in \overline{\mathcal{X}}. \tag{59}$$

That means that after one more multiplication, $\mathbf{x}[N] = \mathbf{A}[N]\mathbf{x}[N-1]$, we finally have $\|\mathbf{x}[N]\| < 1$. So, the number of time steps needed to guarantee contraction of the worst-case unit-length state vector is $\mu = N$, therefore we need to use Criterion 1(N) for an order-N power-normalized ladder filter. To visualize this:

$$\begin{array}{c|c|c} n & \mathbf{x}[n+1] = \left(\prod_{\eta=1}^{n}\mathbf{A}[\eta]\right)\mathbf{x}[0] & \|\mathbf{x}[n]\| \\ \hline 0 & [1 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0]^{\mathrm{T}} & 1 \\ 1 & [-\gamma_1[n] \quad \kappa_1[n] \quad 0 \quad \cdots \quad 0 \quad 0]^{\mathrm{T}} & 1 \\ \vdots & \vdots & \vdots \\ N-1 & [* \quad * \quad * \quad \cdots \quad * \quad *]^{\mathrm{T}} & 1 \\ N & \underbrace{[\dagger \quad \dagger \quad \dagger \quad \cdots \quad \dagger \quad \dagger]^{\mathrm{T}}}_{N} & < 1 \end{array}$$

where again, † represents arbitrary values for a vector that do not have a vector norm of 1, and $*$ represents a vectors that has a vector norm of 1: vectors for which there is no space to print the full expression but which are easily derived.

7. CONCLUSION

Fig 6 provides an implication graph of the Lemmas and Theorems discussed in this paper.

For certain classes of filters, we have given proofs for the number of time steps we need to prove stability using Criterion 1(μ). For filters involving length-N delay lines, we have $N \leq \mu$. This holds with equality for certain filters, such as delay lines of length-N and order-N power-normalized ladder filters. However, we have also seen cases, such as the 2nd-order SVF, where $N < \mu$; this filter only has delay lines of length 1, yet requires the use of Criterion 1(2). Future work should develop a rigorous way of predicting the minimum required μ (not only the lower bound on that minimum) for the class of all time-varying digital filters.

Beyond proving stability of existing filter designs, an application of our findings would be designing new stable time-varying audio filters, e.g., by using our theorem to choose appropriate similarity transforms or other ways of adjusting a filter realization.

Another avenue for future work would be to combine our insights about the product of state matrices over multiple time steps with a similarity transform, to extend Laroche's *Criterion 2*.

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