ON THE EQUIVALENCE OF INTEGRATOR- AND DIFFERENTIATOR-BASED CONTINUOUS- AND DISCRETE-TIME SYSTEMS

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ABSTRACT

The article performs a generic comparison of integrator- and differentiator based continuous-time systems as well as their discrete-time models, aiming to answer the reoccurring question in the music DSP community of whether there are any benefits in using differentiators instead of conventionally employed integrators. It is found that both kinds of models are practically equivalent, but there are certain reservations about differentiator based models.

1. INTRODUCTION

Discrete-time block diagrams are commonly used in DSP. Their most common type of fundamental memory elements are z^{-1} (unit delay) blocks. The commonly known direct filter forms I and II, including their transposed versions, are built around z^{-1} blocks.

Certain systems are however expressed in terms of larger-scale blocks, which are not fundamental. For example, the Moog ladder filter is commonly understood as a feedback loop around a series of four 1-pole lowpass filters [1]. In comparison, the state-variable filter (SVF) is understood in terms of integrators [2]. An integrator also provides a building block of a particular 1-pole lowpass design, known as the "leaky integrator" (Fig. 1).

Integration is fundamentally a continuous-time phenomenon, thus Fig. 1 can be understood as a continuous-time block diagram, and so can be the block diagram of an SVF. In fact, continuous time block diagrams are commonly used alongside discrete time block diagrams in control theory. In the DSP field their use is traditionally somewhat less common, but can be encountered e.g. in the area of virtual analog filters ([3], [4]), where they provide a visual expression of differential equations describing the system of interest. In continuous-time block diagrams integrators take the role of fundamental memory elements. Since the transfer function of an integrator, expressed in terms of Laplace transform, is $H(s) = 1/s$, the integrators are often notated as $1/s$ or s^{-1} blocks.

The idea of using differentiators instead of or in addition to integrators when designing virtual analog systems has come up regularly in the online music DSP discourse, as well as in some academic works [5]. Whilst this topic has been touched upon in existing system theory [6], this motivated the idea of attempting a deeper discussion of the topic.

In terms of equations, the use of differentiation instead of integration is just a notational difference. However in continuous-time block diagrams this already raises some questions. When these are further converted to discrete time, there is immediately a question, whether the discrete-time systems obtained in two such ways (using an integrator- or a differentiator based continuous-time prototype) are equivalent.

In this paper we shall take a detailed look into the applicability of differentiators in continuous-time block diagrams, as well as the equivalence aspects of integrator- and differentiator-based discrete-time models. We are going to show that the continuous-time applicability occurs under a number of reservations, and so does the discrete-time equivalence.

In Sec. 2 we discuss differentiation in continuous-time diagrams. In Sec. 3 we discuss the discrete-time differentiator-based one-pole. In Sec. 4 we generalize to arbitrary linear systems. In Sec. 5 we generalize to nonlinear systems, mixed integration and differentiation, and arbitrary integration schemes.

2. CONTINUOUS-TIME DIFFERENTIATION

2.1. Continuous-time 1-pole

To illustrate the relationship between integrator- and differentiator-based continuous-time systems we use a 1-pole lowpass filter. The integrator-based version of this filter is shown as a block diagram in Fig. 1, the detailed explanation can be found in [3] Ch.2.

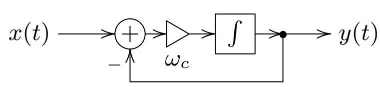

Figure 1: *Integrator-based 1-pole lowpass filter.*

The block diagram in Fig. 1 corresponds to the differential equation:

$$\dot{y}(t) = \omega_c(t) \cdot (x(t) - y(t)) \tag{1}$$

where $\omega_c(t)$ is the (potentially varying) cutoff frequency. This can be more easily seen by integrating both parts of equation (1) obtaining

$$y(t) = \int_0^t \omega_c(\tau)\,(x(\tau) - y(\tau))\,\mathrm{d}\tau \tag{2}$$

where we assume a zero initial state at $t = 0$.

In order to obtain the differentiator-based counterpart of Fig. 1 we need to resolve the equation (1) with respect to $y(t)$, which gives

$$y(t) = x(t) - \frac{1}{\omega_c(t)}\dot{y}(t) \tag{3}$$

Equation (3) can be expressed in the block-diagram form as shown in Fig. 2.

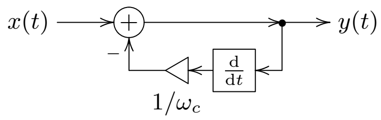

Figure 2: *Differentiator-based 1-pole lowpass filter.*

Under the condition $\omega_c \neq 0$, equations (1) and (3) are equivalent, and so, upon the first look, must be the systems in Figs. 1 and 2 respectively.

2.2. Natural causality

There is however one important difference between equations and block diagrams in that the latter also express causality (in the form of directional connections: the upstream output defines the downstream input, not vice versa) [7]. This implied causality may also have been the reason why delay-less loops were de-facto almost taboo in DSP block diagrams for a long time. In comparison, equations are typically understood as simply statements that two sides are equal, without implying which side defines the other, except for fully explicitly resolved equations, which may be often understood as the RHS defining the LHS, but do not necessarily imply that.

Now, even though equations are not necessarily explicitly containing any causality information, the associated causality can be often derived from the context. For example, Ohm's law $U = IR$ can describe a situation where the voltage defines the current or vice versa. If a load is connected to a voltage source, then the current is defined by the voltage, not the other way around. In fact, on a finer time scale the current is not simply equal to U/R. Suppose there is no voltage on a load and respectively no current flowing through it, and then a voltage is suddenly applied. Since the electrons have a mass, it will take a certain amount of time for the electrons to start moving. Respectively, the current will gradually grow until the electrons are finally moving at a speed such that the applied voltage is fully balanced out by the resistance of the media. Thus, the current value given by U/R is not the instantaneous one but rather the limiting one, which is attained after the system stabilizes. Respectively, Ohm's law gives an equation describing the stablized rather than the instantaneous state. Typically we are interested in time scales much larger than the characteristic time of such stabilization and thus Ohm's law provides an acceptable simplification.

Certain equations unequivocally imply causal relationships. E.g. the capacitor equation

$$\dot{q} = \frac{\mathrm{d}}{\mathrm{d}t}(CU) = I \tag{4}$$

actually expresses the fact that the capacitor charge $q = CU$ is obtained as the accumulated current:

$$q = CU = \int_0^t I(\tau)\,\mathrm{d}\tau \tag{5}$$

that is, the capacitor charge is the result of accumulating the current applied to the capacitor. Similarly the inductor current is the result of accumulating the applied voltage.

Thus, in the causal sense, the capacitor and the inductor are behaving as integrators rather than differentiators. It would be appropriate therefore to represent them as integrators after converting the circuit schematics to a (continuous-time) block diagram, and conversely, representing them as differentiators would look like ignoring their natural causality.

Since the capacitor and the inductor equations are the only differential equations occurring in descriptions of typical simple circuits, it follows that their block diagram representations would naturally contain integrators, whereas differentiator forms would be "causally unnatural".

2.3. Differentiator lookahead

The formal definition of a derivative involves looking at the values before and after the point of interest. Thus, differentiation (with respect to time) implies (if only an infinitely small) lookahead, and hence by definition is not causal.

This raises an interesting philosophical question - Maxwell's equations contain time derivatives of the strengths of electric and magnetic fields, and, by the nature of those equations, this differentiation must be causal. Without trying to get into further philosophical depths, we would be happy just with pointing out this difficulty, as well as with noticing that we could at least attempt to amend the difficulty by understanding time derivatives as left-hand derivatives.

2.4. Implied bandlimitedness

We are now going to highlight one further not so obvious aspect of block diagrams: the causality of the connections also implies their bandlimitedness. This aspect is typically irrelevant, however becomes somewhat important with instantaneous (delay-less or integrator-less) feedback loops and even more important with differentiators in feedback loops, so that it cannot be really ignored anymore.

The concept of causality involves a time relationship between two events: A must happen *before* B. In discrete-time block diagrams this manifests itself in the order of computations implied by the directions of the connections. Discrete-time block diagrams conceptually operate in an ideal world, where the time is "frozen" during each sample, however within this "frozen time" there is a "nested" time scale, on which the computations occurring within this sample are ordered.

This nested time scale becomes crucial in causal treatment of delay-less feedback loops. One way of understanding those, proposed in [8], is to treat this nested scale as also discrete. For delay-less feedback loops this conceptually implies introducing an "infinitely short" delay into the loop and running such loop infinitely many times. In case the iterations converge, the limiting value is taken as the value produced during the "primary time scale sample". This approach however converges only for $|g| < 1$ where g is the instantaneous loop gain.

An approach which converges $\forall g < 1$ (or $\forall \operatorname{Re} g < 1$ if $g \in \mathbb{C}$) is to treat the nested time scale as a continuous time one.[1] The details can be found in [3] Sec.3.13. This can be seen as introducing a "very high cutoff" causal lowpass filter instead of introducing a "very short" delay, thereby bandlimiting the loop. The same approach can be also applied to continuous time block diagrams, where instead of a "nested time scale" it would be more appropriate to talk of a "finer time scale". Notice that this finer

[1] $g \leq -1$ can particularly occur in trapezoidal integration at high cutoff values. The divergence at $g \geq 1$ corresponds to leaving the limits of the integration scheme's applicability.

time scale idea is exactly the same one which we introduced in the discussion of Ohm's law in Sec.2.2, thus it has a 1:1 correspondence to the behavior of real world systems.

For a loop containing an integrator this bandlimiting is irrelevant, because the loop gain of such loop at high frequencies is very small anyway (from the time-domain perspective, an integrator cannot change its value instantaneously, as long as its input is finite). In pure algebraic instantaneous loops the gain is frequency-independent, therefore it's no longer small at high frequencies. Bandlimiting of such loops simply allows their causal treatment. For loops containing differentiators (and no integrators) the gain indefinitely grows with frequency. A bandlimiting filter with a stronger rolloff, and respectively stronger phase shift in the transition and stop bands, will be therefore required, and it's difficult to say whether it will be always possible to ignore the arising artifacts. Attempting a causal time-domain analysis of such instantaneous feedback loop is also problematic as we will need to infinitely differentiate rather than infinitely integrate the input signal, which creates difficulties with any kind of discontinuities in the signal or its derivatives.

Thus, there are certain difficulties with continuous-time block diagrams containing differentiators, and especially containing differentiators in integrator-less loops. Ultimately all those difficulties originate from the causal interpretation of continuous-time block diagrams. The switch from integrators (s^{-1} blocks) to differentiators (s blocks) can be compared to an attempt to use z blocks instead of z^{-1} blocks in discrete-time block diagrams, which would be a clear violation of causality.

3. DISCRETE-TIME 1-POLE

The situation looks notably different in the discrete-time case. The main reasons for this are probably the causality of some of commonly used discrete-time differentiation schemes as well as the inherently limited bandwidth of the media. Respectively, with differentiators in feedback loops we are going to get the usual (more or less) delay-less feedback loops, like the ones occurring in implicit integration methods. We will start the discussion by converting the differentiator-based 1-pole (Fig. 2) to discrete time and compare the conversion result to the conversion result of the integrator-based 1-pole (Fig. 1).

3.1. Integrator-based discrete-time 1-pole

First, let's establish the reference by converting the integrator based 1-pole to discrete time. The trapezoidal integration scheme is a popular choice. In the equation form we would need to replace the integration in equation (2) by the trapezoidal integration:

$$y[n] = y[n-1] + \frac{x[n] + x[n-1]}{2} \tag{6}$$

(where we assume the unit sampling period and where x and y stand for the integrator's input and output signals respectively). In block-diagram terms we can just replace the integrator block in Fig. 1 by a discrete-time integrator implementing trapezoidal integration. Taken literally, equation (6) implies a direct form I integrator, however we can take any other structure implementing the same discrete-time transfer function.[2] A popular topology for the trapezoidal integrator is the transposed direct form II (Fig. 3). Compared to direct form I, this topology uses only a single z^{-1} block, while, compared to direct form II, the value stored in the z^{-1} block's memory is approximately equal to the integrator's output value. See [3] Sec.3.6 for an explicit derivation of Fig. 3.

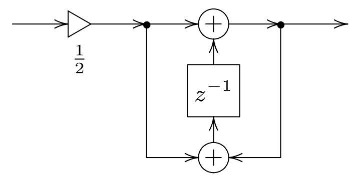

Figure 3: *Transposed direct form II trapezoidal integrator.*

Substituting Fig. 3 into Fig. 1 we obtain the block diagram in Fig. 4.

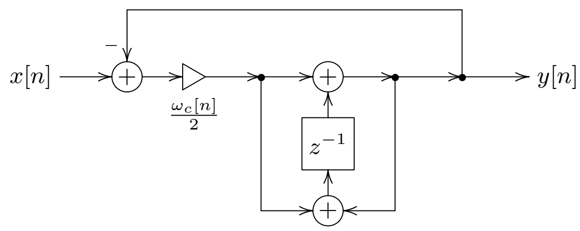

Figure 4: *Integrator-based discrete-time 1-pole lowpass.*

Let $u[n]$ denote the input of the z^{-1} block in Fig. 4 at time moment n. Respectively, the output of the z^{-1} block at the same time is $u[n-1]$. Effectively $u[n-1]$ is the system state at the beginning of the n-th sample, and $u[n]$ is the system state at the end of the n-th sample.

Writing the equation for the main feedback loop we obtain

$$y[n] = \frac{\omega_c[n]}{2}(x[n] - y[n]) + u[n-1] \tag{7}$$

Following the discussion given in Sec. 2.4 the convergence condition for the instantaneous loop is that the loop's gain must be less than unity. By (7) the instantaneous loop gain (which is equal to the coefficient at $y[n]$ in the RHS) is $-\omega_c[n]/2$ and the loop converges as long as $-\omega_c/2 < 1$, that is $\omega_c > -2$ (smaller values of ω_c break the applicability limits of trapezoidal integration).

[2] Any integrator structure implementing the same transfer function is going to result in the same behaviour of the entire system. This fact might be obvious with explicit integration schemes, where the integrator's output depends only on past input, and is also obvious in the LTI case, where we can reason about the entire system in terms of transfer functions. Now consider the case of an implicit scheme integrator in a non-LTI case. The integrator itself is still LTI and can be associated with its transfer function $H(z)$. Since the scheme is implicit, $H(z)$ is a nonstrictly proper rational function of z. Rewrite it as a strictly proper function plus a constant term: $H(z) = G(z) + g$. The term $G(z)$ will define the explicit part of the scheme and at each moment its output is determined solely by the past history. The term g defines the integrator's feedforward delay-less path's gain, which is solely responsible for making the scheme implicit. Since $H(z)$ is always the same, g is also always the same, and at each sample we will be always arriving at the same implicit equation with the same parameter values, thereby producing the same output and the same new system state, no matter what the specific integrator structure is.

Resolving with respect to the output signal $y[n]$ we obtain

$$y[n] = \left(1 + \frac{\omega_c[n]}{2}\right)^{-1} \left(\frac{\omega_c[n]}{2} x[n] + u[n-1]\right) \tag{8}$$

The state update equation is obtained by writing the equation for the trapezoidal integrator part:

$$u[n] = 2\frac{\omega_c[n]}{2}(x[n] - y[n]) + u[n-1] \tag{9}$$

We are going to use these equations as the reference to compare the differentiator-based discrete-time lowpass to.

3.2. Differentiator-based discrete-time 1-pole

In order to obtain a differentiator-based discrete-time 1-pole we need to substitute a "trapezoidal differentiator" into Fig. 2. The difference equation for the trapezoidal differentiator can be obtained by inverting the equation (6), yielding

$$y[n] = 2(x[n] - x[n-1]) - y[n-1] \tag{10}$$

(where we also swapped x and y notation, so that x is the differentiator's input and y is the differentiator's output).

We could now use some standard topology to implement equation (10), but we also can simply do a minor tweak to the integrator in Fig. 3 instead. Consider the transfer function of a trapezoidal integrator. Regardless of the implementation topology, the transfer function is

$$H(z) = \frac{1}{2} \cdot \frac{z+1}{z-1} = \frac{1}{2} \cdot \frac{1+z^{-1}}{1-z^{-1}} \tag{11}$$

(which is simultaneously the substitution formula for s^{-1} in the bilinear transform). Since differentiation is reciprocal to integration in the s-domain, the transfer function of a trapezoidal differentiator must be the reciprocal of equation (11):

$$H(z) = 2\frac{z-1}{z+1} = 2\frac{1-z^{-1}}{1+z^{-1}} \tag{12}$$

Comparing equation (12) to equation (11), we notice that the only difference, besides the reciprocated gain, is that the sign of z^{-1} has been flipped everywhere. We can therefore simply take the block diagram in Fig. 3 and invert the sign of the signal before or after the z^{-1} block, obtaining the block diagram in Fig. 5, where we also remembered to reciprocate the gain. We also moved the gain from the input to the output position, this is purely a convenience transformation, which we are going to appreciate a bit later (it doesn't affect the result, but it will be simpler to compare the result to the reference).

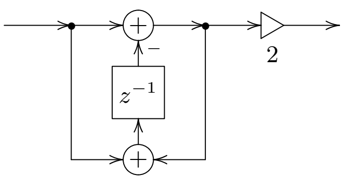

Figure 5: *Trapezoidal differentiator.*

Substituting Fig. 5 for the differentiator in Fig. 2 we obtain the block diagram in Fig. 6.

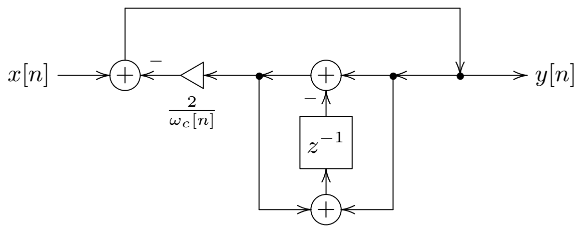

Figure 6: *Differentiator-based discrete-time 1-pole lowpass.*

Again, let $u[n]$ denote the input of the z^{-1} block in Fig. 6 at time moment n. Writing the equation for the main feedback loop we obtain

$$y[n] = x[n] - \frac{2}{\omega_c[n]}(y[n] - u[n-1]) \tag{13}$$

Resolving for $y[n]$ we get

$$y[n] = \left(1 + \frac{2}{\omega_c[n]}\right)^{-1} \left(x[n] + \frac{2}{\omega_c[n]} u[n-1]\right) \tag{14}$$

By multiplying the expressions in each pair of parentheses in (14) by $\omega_c[n]/2$ we obtain equation (8). That is, the main feedback equations for the integrator- and differentiator-based 1-poles are equivalent, at least for $\omega_c \neq 0$.

Now let's write the state update equation by writing the equation for the differentiator part of Fig. 6:

$$u[n] = 2y[n] - u[n-1] \tag{15}$$

On the other hand, we have just established the equivalence of (14) and (8). Respectively (13) and (7) are also equivalent and we can substitute (7) into (15):

$$u[n] = 2\left(\frac{\omega_c[n]}{2}(x[n] - y[n]) + u[n-1]\right) - u[n-1] \tag{16}$$

It's easy to see that (16) is equivalent to (9).

What is however not identical between the two versions are the convergence conditions of the instantaneous feedback loop. As we should remember, the convergence occurs when the instantaneous loop gain is less than unity. By (13), the instantaneous loop gain for Fig. 6 is $-2/\omega_c$, thus the convergence condition is $-2/\omega_c < 1$, that is $1/\omega_c > -\frac{1}{2}$, that is either $\omega_c > 0$ or $\omega_c < -2$. The instantaneous loop therefore fails to converge for $-2 \leq \omega_c \leq 0$. This is different from the convergence condition of the instantaneous loop in Fig. 4. The formal solution (8), (9) is the same for both filters, however the applicability range is different. It is unlikely, though, that the solution obtained for $\omega_c < -2$ is going to make much sense, since this is the range where trapezoidal integration doesn't work well,[3] and the solution formula is exactly the same for both integration and differentiation.

Thus, the differentiator-based 1-pole lowpass in Fig. 6 is fully equivalent to the integrator-based 1-pole lowpass in Fig. 4 only in the range $\omega_c > 0$. For other values of ω_c the equivalence is at least questionable. Now was this equivalence occurring for $\omega_c > 0$ pure luck or is this a general property?

[3] In continuous time the output is exponentially growing, in discrete time the output is exponentially growing and alternating sign with each sample, caused by a pole at $z = (1 - \omega_c/2)/(1 + \omega_c/2) < -1$.

4. ARBITRARY LINEAR SYSTEMS

4.1. Continuous-time prototypes

Let's now suppose we are given an arbitrary integrator-based block-diagram of a linear differential system. Pretty much any such block diagram[4] can be transformed into the generic state space block diagram form (Fig. 7).

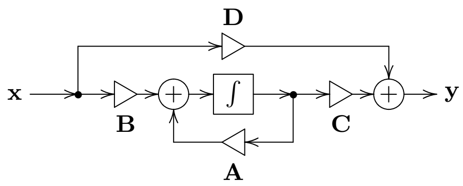

Figure 7: *A generic state-space block-diagram. All signals are vectors (of potentially different dimensions) and all gains are matrices.*

As we did with the 1-pole in Sec.3, we would like to obtain a corresponding differentiator-based block diagram. Following the same steps as in Sec.3, we write the equations describing the system in Fig. 7:

$$\begin{aligned}\dot{\mathbf{u}}(t) &= \mathbf{A}(t)\mathbf{u}(t) + \mathbf{B}(t)\mathbf{x}(t)\\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{u}(t) + \mathbf{D}(t)\mathbf{x}(t)\end{aligned} \tag{17}$$

where $\mathbf{u}$ is the state vector. Just for the reference we can also rewrite equations (17) in the explicit integrating form, similar to (2):

$$\begin{aligned}\mathbf{u}(t) &= \int_0^t \left(\mathbf{A}(\tau)\mathbf{u}(\tau) + \mathbf{B}(\tau)\mathbf{x}(\tau)\right)\,\mathrm{d}\tau\\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{u}(t) + \mathbf{D}(t)\mathbf{x}(t)\end{aligned} \tag{18}$$

To obtain the equivalent equations in an explicit differentiating form, we introduce $\mathbf{u}' = \dot{\mathbf{u}}$ as the new state variable.[5] By the first of equations (17)

$$\mathbf{u}(t) = \mathbf{A}^{-1}(t)\mathbf{u}'(t) - \mathbf{A}^{-1}(t)\mathbf{B}(t)\mathbf{x}(t) \tag{19}$$

and it follows that

$$\mathbf{u}'(t) = \dot{\mathbf{u}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{A}^{-1}(t)\mathbf{u}'(t) - \mathbf{A}^{-1}(t)\mathbf{B}(t)\mathbf{x}(t)\right) \tag{20}$$

while by the second of (17)

$$\mathbf{y}(t) = \mathbf{C}(t)\left(\mathbf{A}^{-1}(t)\mathbf{u}'(t) - \mathbf{A}^{-1}(t)\mathbf{B}(t)\mathbf{x}(t)\right) + \mathbf{D}(t)\mathbf{x}(t) \tag{21}$$

Introducing matrices

$$\begin{aligned}\mathbf{A}'(t) &= \mathbf{A}^{-1}(t)\\ \mathbf{B}'(t) &= -\mathbf{A}^{-1}(t)\mathbf{B}(t)\\ \mathbf{C}'(t) &= \mathbf{C}(t)\mathbf{A}^{-1}(t)\\ \mathbf{D}'(t) &= \mathbf{D}(t) - \mathbf{C}(t)\mathbf{A}^{-1}(t)\mathbf{B}(t)\end{aligned} \tag{22}$$

we obtain the equations in the explicit differentiation form:

$$\begin{aligned}\mathbf{u}'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{A}'(t)\mathbf{u}'(t) + \mathbf{B}'(t)\mathbf{x}(t)\right)\\ \mathbf{y}(t) &= \mathbf{C}'(t)\mathbf{u}'(t) + \mathbf{D}'(t)\mathbf{x}(t)\end{aligned} \tag{23}$$

The block diagram in Fig. 8 implied by (23) is fully analogous to the one in Fig. 7.

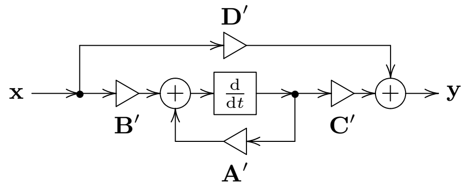

Figure 8: *A generic state-space block-diagram in the differentiator-based form.*

Conversely, given a block diagram in the form of Fig. 8, we can invert (22) to obtain the coefficients of the representation in the form of Fig. 7:

$$\begin{aligned}\mathbf{A}(t) &= \mathbf{A}'^{-1}(t)\\ \mathbf{B}(t) &= -\mathbf{A}'^{-1}(t)\mathbf{B}'(t)\\ \mathbf{C}(t) &= \mathbf{C}'(t)\mathbf{A}'^{-1}(t)\\ \mathbf{D}(t) &= \mathbf{D}'(t) - \mathbf{C}'(t)\mathbf{A}'^{-1}(t)\mathbf{B}'(t)\end{aligned} \tag{24}$$

It follows that the block diagram in Fig. 7 can be equivalently represented by the one in Fig. 8, provided $\mathbf{A}(t)$ is nonsingular at all t. Conversely, the block diagram in Fig. 7 can be equivalently represented by the one in Fig. 8, provided $\mathbf{A}'(t)$ is nonsingular at all t. Thus, under the reservations discussed in Sec.2, we can consider non-singular integrator- and differentiator-based state-space forms as equivalent.[6]

4.2. Integrator discretization

We now wish to find out if this equivalence is preserved by time discretization. For the sake of limited space we are going to use the direct form I integration and differentiation topologies (Figs. 9 and 10) which are directly expressing the formulas (6) and (10), thereby simplifying the algebraic transformations. The analysis of the general case, which includes the use of arbitrary topologies is done in Sec. 5.

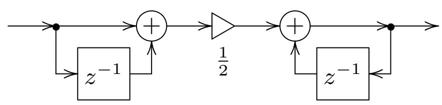

Figure 9: *Direct form I trapezoidal integrator.*

[4] The exceptions can occur in an exotic case of non-converging integrator-less feedback loops with gains $g \geq 1$.

[5] The prime sign here doesn't denote a derivative, $\mathbf{u}'$ is simply a different variable than $\mathbf{u}$.

[6] Notably, a singular $\mathbf{A}(t)$ implies that one of the eigenvalues of $\mathbf{A}(t)$ (which are the system poles) is zero, thus the system performs pure integration along the dimension of the respective eigenvector. Clearly, pure integration cannot be equivalently expressed via differentiation. Conversely, a singular $\mathbf{A}'(t)$ would correspond to pure differentiation done along some eigenvector of $\mathbf{A}'(t)$, which cannot be expressed via integration. As feedforward integration is rarely used in music DSP, this case is not important.

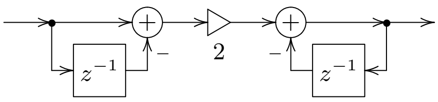

Figure 10: *Direct form I trapezoidal differentiator.*

Let's first discretize the integrator form and use the result as the reference. Algebraically, substituting Fig. 9 into Fig. 7 means substitution of equation (6) into (18), yielding

$$\begin{aligned}\mathbf{u}[n] &= \mathbf{u}[n-1] + \frac{1}{2}(\mathbf{A}[n]\mathbf{u}[n] + \mathbf{B}[n]\mathbf{x}[n] \\ &\qquad + \mathbf{A}[n-1]\mathbf{u}[n-1] + \mathbf{B}[n-1]\mathbf{x}[n-1]) \\ \mathbf{y}[n] &= \mathbf{C}[n]\mathbf{u}[n] + \mathbf{D}[n]\mathbf{x}[n]\end{aligned} \tag{25}$$

where the second of equations (18) is simply discretized naively. The first of the equations (25) can be resolved in respect to $\mathbf{u}[n]$ yielding

$$\begin{aligned}\mathbf{u}[n] &= \left(1 - \frac{\mathbf{A}[n]}{2}\right)^{-1} \left(\left(1 + \frac{\mathbf{A}[n-1]}{2}\right) \mathbf{u}[n-1] \right. \\ &\qquad \left. + \frac{1}{2}\Big(\mathbf{B}[n]\mathbf{x}[n] + \mathbf{B}[n-1]\mathbf{x}[n-1]\Big)\right) \\ \mathbf{y}[n] &= \mathbf{C}[n]\mathbf{u}[n] + \mathbf{D}[n]\mathbf{x}[n]\end{aligned} \tag{26}$$

4.3. Differentiator discretization

Substitution of Fig. 10 into Fig. 8 algebraically means substitution of equation (10) into (23), yielding

$$\begin{aligned}\mathbf{u}'[n] &= 2\Big(\mathbf{A}'[n]\mathbf{u}'[n] + \mathbf{B}'[n]\mathbf{x}[n] - \mathbf{A}'[n-1]\mathbf{u}'[n-1] \\ &\qquad - \mathbf{B}'[n-1]\mathbf{x}[n-1]\Big) - \mathbf{u}'[n-1] \\ \tilde{\mathbf{y}}[n] &= \mathbf{C}'[n]\mathbf{u}'[n] + \mathbf{D}'[n]\mathbf{x}[n]\end{aligned} \tag{27}$$

We denoted the output signal as $\tilde{\mathbf{y}}[n]$ because at this moment we don't know yet whether the output signals of (25) and (27) are identical.

Resolving the first equation in respect to $\mathbf{u}'[n]$ we obtain:

$$\begin{aligned}\mathbf{u}'[n] &= \Big(1 - 2\mathbf{A}'[n]\Big)^{-1}\Big(2\mathbf{B}'[n]\mathbf{x}[n] - 2\mathbf{B}'[n-1]\mathbf{x}[n-1] \\ &\qquad - \Big(1 + 2\mathbf{A}'[n-1]\Big)\mathbf{u}'[n-1]\Big) \\ \tilde{\mathbf{y}}[n] &= \mathbf{C}'[n]\mathbf{u}'[n] + \mathbf{D}'[n]\mathbf{x}[n]\end{aligned} \tag{28}$$

4.4. Equivalence

Expressing the second of the equations (28) in terms of matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ we obtain

$$\tilde{\mathbf{y}}[n] = \mathbf{C}[n]\mathbf{A}^{-1}[n]\mathbf{u}'[n] + \Big(\mathbf{D}[n] - \mathbf{C}[n]\mathbf{A}^{-1}[n]\mathbf{B}[n]\Big)\mathbf{x}[n] \tag{29}$$

We want to check if $\tilde{\mathbf{y}}[n] = \mathbf{y}[n]$. By (29) and (26) this is equivalent to

$$\begin{aligned}\mathbf{C}[n]\mathbf{A}^{-1}[n]\mathbf{u}'[n] + \Big(\mathbf{D}[n] - \mathbf{C}[n]\mathbf{A}^{-1}[n]\mathbf{B}[n]\Big)\mathbf{x}[n] \\ = \mathbf{C}[n]\mathbf{u}[n] + \mathbf{D}[n]\mathbf{x}[n]\end{aligned} \tag{30}$$

or

$$\mathbf{C}[n]\mathbf{A}^{-1}[n]\mathbf{u}'[n] - \mathbf{C}[n]\mathbf{A}^{-1}[n]\mathbf{B}[n]\mathbf{x}[n] = \mathbf{C}[n]\mathbf{u}[n] \tag{31}$$

which holds if

$$\mathbf{A}^{-1}[n]\mathbf{u}'[n] - \mathbf{A}^{-1}[n]\mathbf{B}[n]\mathbf{x}[n] = \mathbf{u}[n] \tag{32}$$

or

$$\mathbf{u}'[n] - \mathbf{B}[n]\mathbf{x}[n] = \mathbf{A}[n]\mathbf{u}[n] \tag{33}$$

or

$$\mathbf{u}'[n] = \mathbf{A}[n]\mathbf{u}[n] + \mathbf{B}[n]\mathbf{x}[n] \tag{34}$$

Let's see if condition (34) holds. Suppose both systems are starting from zero initial states $\mathbf{u}[n] = \mathbf{u}'[n] = 0\ \forall n \le 0$ and receive the same input signal $\mathbf{x}[n]$ starting with $n = 1$, while $\mathbf{x}[n] = 0\ \forall n \le 0$. Apparently, (34) holds $\forall n \le 0$.[7] We are going to show by induction that (34) also holds $\forall n > 0$.

Suppose we are at a sample $n > 0$ and suppose (34) holds for all previous samples. Taking the first of the equations of (28) and expressing it in terms of matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, we obtain

$$\begin{aligned}\mathbf{u}'[n] = \Big(1 - 2\mathbf{A}^{-1}[n]\Big)^{-1}\Big(&-2\mathbf{A}^{-1}[n]\mathbf{B}[n]\mathbf{x}[n] \\ &+ 2\mathbf{A}^{-1}[n-1]\mathbf{B}[n-1]\mathbf{x}[n-1] \\ &- \Big(1 + 2\mathbf{A}^{-1}[n-1]\Big)\mathbf{u}'[n-1]\Big)\end{aligned} \tag{35}$$

Since (34) holds at $n-1$ by our assumption, we can substitute it for $\mathbf{u}'[n-1]$, obtaining

$$\begin{aligned}\mathbf{u}'[n] &= \Big(1 - 2\mathbf{A}^{-1}[n]\Big)^{-1}\Big(-2\mathbf{A}^{-1}[n]\mathbf{B}[n]\mathbf{x}[n] \\ &\quad + 2\mathbf{A}^{-1}[n-1]\mathbf{B}[n-1]\mathbf{x}[n-1] \\ &\quad - \Big(1 + 2\mathbf{A}^{-1}[n-1]\Big)\Big(\mathbf{A}[n-1]\mathbf{u}[n-1] \\ &\qquad\qquad + \mathbf{B}[n-1]\mathbf{x}[n-1]\Big)\Big) \\ &= \Big(1 - 2\mathbf{A}^{-1}[n]\Big)^{-1}\Big(-2\mathbf{A}^{-1}[n]\mathbf{B}[n]\mathbf{x}[n] \\ &\quad - \mathbf{B}[n-1]\mathbf{x}[n-1] - \Big(2 + \mathbf{A}[n-1]\Big)\mathbf{u}[n-1]\Big) \\ &= \left(\frac{\mathbf{A}[n]}{2} - 1\right)^{-1}\frac{\mathbf{A}[n]}{2}\Big(-2\mathbf{A}^{-1}[n]\mathbf{B}[n]\mathbf{x}[n] \\ &\quad - \mathbf{B}[n-1]\mathbf{x}[n-1] - \Big(2 + \mathbf{A}[n-1]\Big)\mathbf{u}[n-1]\Big) \\ &= \left(1 - \frac{\mathbf{A}[n]}{2}\right)^{-1}\left(\mathbf{B}[n]\mathbf{x}[n] + \frac{\mathbf{A}[n]}{2}\mathbf{B}[n-1]\mathbf{x}[n-1] \right. \\ &\quad \left. + \mathbf{A}[n]\left(1 + \frac{\mathbf{A}[n-1]}{2}\right)\mathbf{u}[n-1]\right)\end{aligned} \tag{36}$$

On the other hand, taking the RHS from the first equation of (26)

[7] In principle we can assume any other initial conditions, as long as they satisfy (34) $\forall n \le 0$.

and substituting it for $\mathbf{u}[n]$ into the RHS of (34) we obtain

$$\begin{aligned}\mathbf{A}[n]\mathbf{u}[n]+\mathbf{B}[n]\mathbf{x}[n] &= \mathbf{A}[n]\left(1-\frac{\mathbf{A}[n]}{2}\right)^{-1} \\ &\times\Bigg(\left(1+\frac{\mathbf{A}[n]}{2}\right)\mathbf{u}[n-1] \\ &\quad+\frac{1}{2}\Big(\mathbf{B}[n]\mathbf{x}[n]+\mathbf{B}[n-1]\mathbf{x}[n-1]\Big)\Bigg) \\ &+\mathbf{B}[n]\mathbf{x}[n] \\ &=\left(1-\frac{\mathbf{A}[n]}{2}\right)^{-1}\Bigg(\mathbf{A}[n]\left(1+\frac{\mathbf{A}[n]}{2}\right)\mathbf{u}[n-1] \\ &\quad+\frac{\mathbf{A}[n]}{2}\Big(\mathbf{B}[n]\mathbf{x}[n]+\mathbf{B}[n-1]\mathbf{x}[n-1]\Big)\Bigg) \\ &+\mathbf{B}[n]\mathbf{x}[n] \\ &=\left(1-\frac{\mathbf{A}[n]}{2}\right)^{-1}\Bigg(\mathbf{A}[n]\left(1+\frac{\mathbf{A}[n]}{2}\right)\mathbf{u}[n-1] \\ &\quad+\frac{\mathbf{A}[n]}{2}\Big(\mathbf{B}[n]\mathbf{x}[n]+\mathbf{B}[n-1]\mathbf{x}[n-1]\Big) \\ &+\left(1-\frac{\mathbf{A}[n]}{2}\right)\mathbf{B}[n]\mathbf{x}[n]\Bigg) \\ &=\left(1-\frac{\mathbf{A}[n]}{2}\right)^{-1}\Bigg(\mathbf{A}[n]\left(1+\frac{\mathbf{A}[n]}{2}\right)\mathbf{u}[n-1] \\ &\quad+\frac{\mathbf{A}[n]}{2}\mathbf{B}[n-1]\mathbf{x}[n-1]+\mathbf{B}[n]\mathbf{x}[n]\Bigg)\end{aligned} \tag{37}$$

where we used the fact that matrices $\mathbf{A}$ and $(1-\mathbf{A}/2)^{-1}$ commute. Now (36) and (37) provide expressions for the LHS and RHS of (34) respectively. Since both expressions are identical, (34) holds at sample n and $\tilde{\mathbf{y}}[n]=\mathbf{y}[n]$. By induction, $\tilde{\mathbf{y}}=\mathbf{y}$ holds at any $n>0$ and therefore the formal solutions are 100% equivalent.

There is, however, again a reservation about instantaneous loop convergence. By interpreting $\mathbf{u}[n]$ in the first of equations (25) in the eigenbasis of $\mathbf{A}[n]$ we notice that the instantaneous loop gains associated with each of the respective dimensions are the eigenvalues of $\mathbf{A}[n]/2$. On the other hand, the instantaneous loop gains in (27) are the eigenvalues of $2\mathbf{A}'[n]=2\mathbf{A}^{-1}[n]$, which are the reciprocals of the eigenvalues of $\mathbf{A}[n]/2$. The instantaneous loop convergence is therefore identical only for eigenvalues in the left semiplane (which correspond to stable filters).[8]

5. THE FULLY GENERAL CASE

We would have liked to perform the generalizations of the just obtained results step by step, where the generalizations include mixing of integrators and differentiators within one system, systems with nonlinear waveshapers and arbitrary (but still mutually inverse) discrete-time integration and differentiation schemes. However for the sake of limited space we are going to do multiple generalization steps at once, leaving it to the reader(s) to fill in the gaps, if desired.

[8] The discrete-time poles are at $z=(1+\lambda_i/2)/(1-\lambda_i/2)$ where $|\lambda_i-\mathbf{A}|=0$, so that alternating-sign growing exponential output occurs for $\operatorname{Re}\lambda_i>2$. The divergence of the discrete-time instantaneous loop occurs at $\operatorname{Re}\lambda_i\ge 2$ in the integrator case and at $0\le\operatorname{Re}\lambda_i\le 2$ in the differentiator case.

5.1. Continuous-time prototypes

Let's split the state vector $\mathbf{u}$ in equation (18) into two vectors which we denote $\mathbf{u}$ and $\mathbf{v}$. The $\mathbf{u}$ components will be still obtained by integration, while $\mathbf{v}$ components will be obtained either by integration (in our reference system) or by differentiation.

Our reference (explicitly integrating) system is therefore

$$\begin{aligned}\mathbf{u}(t)&=\int_0^t F(\mathbf{u}(\tau),\mathbf{v}(\tau),\mathbf{x}(\tau),\tau)\,\mathrm{d}\tau\\ \mathbf{v}(t)&=\int_0^t G(\mathbf{v}(\tau),\mathbf{u}(\tau),\mathbf{x}(\tau),\tau)\,\mathrm{d}\tau\\ \mathbf{y}(t)&=H(\mathbf{u}(t),\mathbf{v}(t),\mathbf{x}(t),t)\end{aligned} \tag{38}$$

To obtain the mixed (integrating and differentiating) form, introduce the new variable $\mathbf{v}'=\dot{\mathbf{v}}$. By (38):

$$\mathbf{v}'(t)=\dot{\mathbf{v}}(t)=G(\mathbf{v}(t),\mathbf{u}(t),\mathbf{x}(t),t) \tag{39}$$

$$\mathbf{v}(t)=G^{-1}(\mathbf{v}'(t),\mathbf{u}(t),\mathbf{x}(t),t) \tag{40}$$

where the inversion of G is done in respect to the first argument only, therefore the remaining arguments are considered as function parameters. Introducing new function $G'=G^{-1}$

$$\mathbf{v}(t)=G'(\mathbf{v}'(t),\mathbf{u}(t),\mathbf{x}(t),t) \tag{41}$$

Differentiating (41) in respect to t:

$$\mathbf{v}'(t)=\dot{\mathbf{v}}(t)=\frac{\mathrm{d}}{\mathrm{d}t}G'(\mathbf{v}'(t),\mathbf{u}(t),\mathbf{x}(t),t) \tag{42}$$

Reexpressing the remaining 2 equations of (38) in terms of $\mathbf{v}'$:

$$\begin{aligned}\mathbf{u}(t)&=\int_0^t F\big(\mathbf{u}(\tau),G'(\mathbf{v}'(\tau),\mathbf{u}(\tau),\mathbf{x}(\tau),\tau),\mathbf{x}(\tau),\tau\big)\,\mathrm{d}\tau\\ &=\int_0^t F'(\mathbf{u}(\tau),\mathbf{v}'(\tau),\mathbf{x}(\tau),\tau)\,\mathrm{d}\tau\end{aligned} \tag{43}$$

$$\begin{aligned}\mathbf{y}(t)&=H\big(\mathbf{u}(t),G'(\mathbf{v}'(t),\mathbf{u}(t),\mathbf{x}(t),t),\mathbf{x}(t),t\big)\\ &=H'(\mathbf{u}(t),\mathbf{v}'(t),\mathbf{x}(t),t)\end{aligned} \tag{44}$$

where we introduced two further functions F' and H'. Ultimately we obtain a mixed system:

$$\begin{aligned}\mathbf{u}(t)&=\int_0^t F'(\mathbf{u}(\tau),\mathbf{v}'(\tau),\mathbf{x}(\tau),\tau)\,\mathrm{d}\tau\\ \mathbf{v}'(t)&=\frac{\mathrm{d}}{\mathrm{d}t}G'(\mathbf{v}'(t),\mathbf{u}(t),\mathbf{x}(t),t)\\ \mathbf{y}(t)&=H'(\mathbf{u}(t),\mathbf{v}'(t),\mathbf{x}(t),t)\end{aligned} \tag{45}$$

which is mathematically equivalent to (38) under the assumption of invertibility of G. Conversely, (45) can be equivalently converted to (38), provided G' is invertible.

5.2. Discrete-time integration and differentiation operators

In order to abstract our discussion from the specific integration and differentiation schemes used, we need to introduce discrete-time integration and differentiation operators.

Let $\mathcal{I}$ be an operator defining the discrete-time integration scheme in question and $\mathcal{D}$ be the corresponding discrete-time differentiation operator. E.g. the trapezoidal integration operator is defined by (6), which in the operator notation takes the form

$$(\mathcal{I}\mathbf{x})[n]=(\mathcal{I}\mathbf{x})[n-1]+\frac{\mathbf{x}[n]+\mathbf{x}[n-1]}{2} \tag{46}$$

meaning that for an input sequence $\mathbf{x}[\cdot]$ the operator $\mathcal{I}$ returns an output sequence $\mathbf{y}[\cdot] = (\mathcal{I}\mathbf{x})[\cdot]$, such that $\mathbf{x}[\cdot]$ and $\mathbf{y}[\cdot]$ satisfy (6). Similarly, the trapezoidal differentiation operator is defined by (10).

In the discussion, which follows, the operators $\mathcal{I}$ and $\mathcal{D}$ can be two arbitrary mutually inverse operators. Formally, their composition should produce an identity operator: $\mathcal{DI} = \mathcal{ID} = 1$.[9] Thus our discussion will be able to accommodate various discrete-time integration schemes, but in principle the operators do not have to be anything close to integration and differentiation. In order to be usable in practice, both schemes need to be causal and implementable with a finite memory amount. The approach is thus restricted to implicit schemes, as the inverse of an explicit scheme is non-causal. In particular, the forward-difference Euler scheme would be thereby excluded, as its discrete-time differentiation is not causal.

5.3. Equivalence

Discretizing (38) we obtain

$$\begin{aligned} \mathbf{u}[\cdot] &= \mathcal{I}F(\mathbf{u}[\cdot], \mathbf{v}[\cdot], \mathbf{x}[\cdot], \cdot) \\ \mathbf{v}[\cdot] &= \mathcal{I}G(\mathbf{v}[\cdot], \mathbf{u}[\cdot], \mathbf{x}[\cdot], \cdot) \\ \mathbf{y}[n] &= H(\mathbf{u}[n], [n], \mathbf{x}[n], n) \end{aligned} \tag{47}$$

while discretizing (45) we obtain

$$\begin{aligned} \tilde{\mathbf{u}}[\cdot] &= \mathcal{I}F'(\tilde{\mathbf{u}}[\cdot], \mathbf{v}'[\cdot], \mathbf{x}[\cdot], \cdot) \\ \mathbf{v}'[\cdot] &= \mathcal{D}G'(\mathbf{v}'[\cdot], \tilde{\mathbf{u}}[\cdot], \mathbf{x}[\cdot], \cdot) \\ \tilde{\mathbf{y}}[n] &= H'(\tilde{\mathbf{u}}[n], \mathbf{v}'[n], \mathbf{x}[n], n) \end{aligned} \tag{48}$$

where we introduced variables $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{y}}$, because we don't know yet whether they will be equal to $\mathbf{u}$ and $\mathbf{y}$, this is what we want to find out.

By the second of the equations (48)

$$\mathcal{I}\mathbf{v}'[\cdot] = G^{-1}(\mathbf{v}'[\cdot], \tilde{\mathbf{u}}[\cdot], \mathbf{x}[\cdot], \cdot) \tag{49}$$

Let $\tilde{\mathbf{v}}[\cdot] = \mathcal{I}\mathbf{v}'[\cdot]$, respectively $\mathbf{v}'[\cdot] = \mathcal{D}\tilde{\mathbf{v}}[\cdot]$ (the idea of the notation $\tilde{\mathbf{v}}$ is that we hope that $\tilde{\mathbf{v}} = \mathbf{v}$). The above equation turns into

$$\tilde{\mathbf{v}}[\cdot] = G^{-1}(\mathbf{v}'[\cdot], \tilde{\mathbf{u}}[\cdot], \mathbf{x}[\cdot], \cdot) \tag{50}$$

and further into

$$\tilde{\mathbf{v}}[\cdot] = G^{-1}(\mathcal{D}\tilde{\mathbf{v}}[\cdot], \tilde{\mathbf{u}}[\cdot], \mathbf{x}[\cdot], \cdot) \tag{51}$$

$$\mathcal{D}\tilde{\mathbf{v}}[\cdot] = G(\tilde{\mathbf{v}}[\cdot], \tilde{\mathbf{u}}[\cdot], \mathbf{x}[\cdot], \cdot) \tag{52}$$

$$\tilde{\mathbf{v}}[\cdot] = \mathcal{I}G(\tilde{\mathbf{v}}[\cdot], \tilde{\mathbf{u}}[\cdot], \mathbf{x}[\cdot], \cdot) \tag{53}$$

On the other hand, by the first of equations (48) and by the introduction of F' and G'

$$\begin{aligned} \tilde{\mathbf{u}}[\cdot] &= \mathcal{I}F'(\tilde{\mathbf{u}}[\cdot], \mathbf{v}'[\cdot], \mathbf{x}[\cdot], \cdot) \\ &= \mathcal{I}F\big(\tilde{\mathbf{u}}[\cdot], G^{-1}(\mathbf{v}'[\cdot], \tilde{\mathbf{u}}[\cdot], \mathbf{x}[\cdot], \cdot), \mathbf{x}[\cdot], \cdot\big) \\ &= \mathcal{I}F\big(\tilde{\mathbf{u}}[\cdot], G^{-1}(\mathcal{D}\tilde{\mathbf{v}}[\cdot], \tilde{\mathbf{u}}[\cdot], \mathbf{x}[\cdot], \cdot), \mathbf{x}[\cdot], \cdot\big) \\ &= \mathcal{I}F(\tilde{\mathbf{u}}[\cdot], \tilde{\mathbf{v}}[\cdot], \mathbf{x}[\cdot], \cdot) \end{aligned} \tag{54}$$

[9] It is not difficult to verify by induction that the operators defined by (6) and (10) are mutually inverse.

Combining (54) and (53) we obtain an equation system

$$\begin{aligned} \tilde{\mathbf{u}}[\cdot] &= \mathcal{I}F(\tilde{\mathbf{u}}[\cdot], \tilde{\mathbf{v}}[\cdot], \mathbf{x}[\cdot], \cdot) \\ \tilde{\mathbf{v}}[\cdot] &= \mathcal{I}G(\tilde{\mathbf{v}}[\cdot], \tilde{\mathbf{u}}[\cdot], \mathbf{x}[\cdot], \cdot) \end{aligned} \tag{55}$$

which is identical to the first two equations of (47). It follows that $\tilde{\mathbf{u}} = \mathbf{u}$ and $\tilde{\mathbf{v}} = \mathbf{v}$ and it remains to show that $\tilde{\mathbf{y}} = \mathbf{y}$.

Taking the third of the equations (48) we obtain

$$\begin{aligned} \tilde{\mathbf{y}}[n] &= H'(\tilde{\mathbf{u}}[n], \mathbf{v}'[n], \mathbf{x}[n], n) \\ &= H\big(\tilde{\mathbf{u}}[n], G^{-1}(\mathbf{v}'[n], \tilde{\mathbf{u}}[n], \mathbf{x}[n], n), \mathbf{x}[n], n\big) \\ &= H(\tilde{\mathbf{u}}[n], \tilde{\mathbf{v}}[n], \mathbf{x}[n], n) \\ &= H(\mathbf{u}[n], \mathbf{v}[n], \mathbf{x}[n], n) = \mathbf{y}[n] \end{aligned} \tag{56}$$

where we used (50) to get rid of $\mathbf{v}'$. Both systems thus have identical output. The instantaneous loop convergence is however still subject to the usual reservations.

6. CONCLUSION

We have shown that continuous-time integrator- and differentiator-based (as well as mixed) systems defined in terms of block diagrams are equivalent, with certain bandlimiting reservations arising from causality aspects implied with block diagrams containing differentiators. In discrete time the equivalence holds as well, under reservations this time occurring with unstable differentiator-based systems. For most practical purposes, integrator- and differentiator based systems can be considered equivalent and there is no immediate benefit in using differentiators.

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