COMPLEMENTARY N-GON WAVES AND SHUFFLED SAMPLES NOISE

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ABSTRACT

This paper introduces complementary n-gon waves and the shuffled samples noise effect. N-gon waves retain angles of the regular polygons and star polygons of which they are derived from in the waveform itself. N-gon waves are researched by the author since 2000 and were introduced to the public at ICMC|SMC in 2014. Complementary n-gon waves consist of an n-gon wave and a complementary angular wave. The complementary angular wave introduced in this paper complements an n-gon wave so that the two waveforms can be used to reconstruct the polygon of which the waveforms were derived from. If it is derived from a star polygon, it is not an n-gon wave and has its own characteristics. Investigations into how geometry, audio, visual and perception are related led to experiments with complementary n-gon waves and a shuffle algorithm. It is possible to reconstruct a digitised geometric shape from its shuffled samples and visualise the geometric shape with shuffled samples noise signals on a digital display device or also, within some limitations, on an oscilloscope in X-Y mode. This paper focuses on the description of discrete complementary n-gon waves and how a Fisher-Yates shuffle algorithm was applied to these waveforms and used to create the shuffled samples noise effect. In addition, some of the timbral and spatial characteristics of complementary n-gon waves and shuffled samples noise are outlined and audiovisual applications of these waveforms briefly discussed.

1. INTRODUCTION

N-gon waves were introduced in the previous paper *N-Gon Waves – Audio Applications of the Geometry of Regular Polygons in the Time Domain* [1]. The reconstruction of the polygons of which the n-gon waves were derived from with the waveforms themselves was not included in the paper. It is not possible to reconstruct these polygons with two n-gon waves, except for n-gon waves derived from a tetragon (i.e. a square). This paper presents a continuation of the n-gon waves topic by introducing complementary n-gon waves that consist of an n-gon wave and a complementary angular wave that can be used to reconstruct the polygon of which the waveforms were derived from. The complementary angular wave is, with the exception of the tetragon wave, not an n-gon wave itself and has its own characteristics. The reconstructed polygons can be visualised with complementary n-gon waves in the same way as Lissajous Figures are visualised on a cathode-ray oscilloscope in X-Y mode. The complementary n-gon waves topic introduced in this paper is enhanced with presenting the application of the Fisher-Yates shuffle algorithm [2] on complementary n-gon waves. Although, applying a shuffle algorithm to digital samples may not appear very spectacular, it is demonstrated in this paper that it can lead to characteristic audiovisual effects.

A characteristic of an n-gon wave is that it retains angles of the regular polygon or star polygon of which the waveform was derived from. As such, some parts of the polygon can be recognised when the waveform is visualised, as can be seen from figure 1. This characteristic distinguishes n-gon waves from other

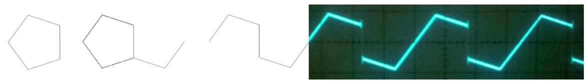

Figure 1: *An n-gon wave is derived from a pentagon. The internal angle of $3\pi/5$ rad of the pentagon is retained in the shape of the wave and in the visualisation of the resulting pentagon wave on a cathode-ray oscilloscope.*

waveforms derived from regular polygons or waveforms that can be used to generate polygonal shapes. Examples for other waveforms related to polygons are the sinusoidal polygonal waveforms by L. Vieira Barbosa [3] and C. Hohnerlein et al. [4], Gielis curves [5] and the polygonal waveforms used in the DIN software by J. Sampath [6]. Some n-gon waves were experimentally constructed with Fourier analysis and resynthesis and briefly discussed in the previous paper [1]. But with the exception of sawtooth, triangle, square and hexagonal waves, n-gon waves do not seem to be covered in common Fourier series literature or synthesis. For instance, n-gon waves are not included in the publication *Fourier Series of Polygons* by Alain Robert [7] or the thesis *The Harmonic Pattern Function: A Mathematical Model Integrating Synthesis of Sound and Graphical Patterns* by L. J. Putnam [8]. Graphical or ornamental waveforms were drawn directly onto the soundtrack part of a filmstrip in the 20s and 30s of the 20th century. For example, by A. Avraamov [9] and O. Fischinger [10]. This could be regarded as, to some degree, similar approach to the transfer of geometric shapes into a waveform, as used with n-gon waves. The relationship between audio and visual in the Oscilloscope Music by J. Fenderson [11] and the OsciStudio software by H. Raber [12] is described by J. Fenderson as "what you see is what you hear" [13]. The description seems to be appropriate, because, in OsciStudio audio signals derived from graphics that are played back through speakers can be synchronously visualised on a cathode-ray oscilloscope in X-Y mode. As complementary n-gon waves reconstruct the polygon of which the waveforms were derived from, this polygon can also be visualised on a digital display device or an oscilloscope in X-Y mode. A prototype of a complementary n-gon waves oscillator VST-plugin was used to investigate this relation of geometry, audio, visual and perception. An audiovisual phenomenon was observed: the same or a very similar shape can be constructed with different waveforms. For instance, waveforms based on amplitude modulated sinusoidals, linear waveforms or Fourier polygon series can all be used to display a regular polygon on a cathode-ray oscilloscope in X-Y mode. Also, they may all have their unique timbres. This observation led to experiments with a Fisher-Yates shuffle algorithm, applied to the sequence of samples of complementary n-gon waves. The rationale behind this approach was to find out whether noise signals could be used to visualise polygons and if the application of the Fisher-Yates shuffle algorithm could be used as an audio effect for complementary n-gon waves.

2. COMPLEMENTARY N-GON WAVES

2.1. Complementary N-gon Waves Definition

An n-gon wave and another angular waveform that complement each other so that they can be used to reconstruct the regular polygon or star polygon of which they were derived from are given the name complementary n-gon waves in this paper. The two waveforms of the complementary n-gon waves are referred to as n-gon wave and complementary angular wave. If the n-gon wave is defined as a function $f(\sin x)$ then the complementary angular wave is the function $f(\cos x)$. The n-gon wave retains angles of the polygon it is derived from in the waveform itself. The angles of the complementary angular wave are orthogonal, but it shares the frequency and the position of the star polygon vertices in the time domain with the n-gon wave. The complementary angular wave derived from a regular polygon is a tetragon wave. However, the complementary angular wave derived from a regular star polygon is not an n-gon wave, as it does not retain angles of the star polygon it is derived from. This is also evident from the slope equations of the n-gon wave (6) and the complementary angular wave (10) and figure 2. Note that the waveform graphics in this paper are inverted so that the they can be attached to a polygon and read clockwise from left to right instead of anticlockwise right to left.

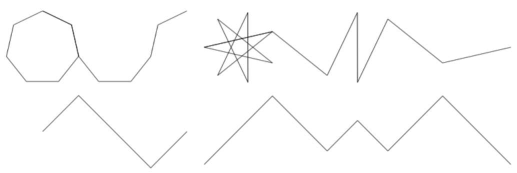

Figure 2: *Complementary n-gon waves: N-gon waves are attached to a reconstructed heptagon and heptagram with Schläfli symbols* $\{7/1\}$ *and* $\{7/3\}$ *of which the two waveforms are derived from. Below the n-gon waves are their complementary angular waves.*

2.2. Complementary N-gon Waves Equations

Discrete complementary n-gon waves can be calculated with the equations presented in this paper. Some of the equations are used to calculate intermediary steps of both the n-gon wave and the complementary n-gon wave and others are only used to calculate one of the two waveforms. The equations are parts of two larger equations that can be used to calculate discrete n-gon waves and their complementary angular waves.

θ is the central angle of the polygon the complementary n-gon waves are derived from. In equation (1), the variable n is the number of vertices of the polygon and the variable q is the polygon density or turning number, they are used in the Schläfli symbol $\{n/q\}$ to define the polygon.

$$\theta = 2\pi q/n \tag{1}$$

In equation (2) the size of the radius r_0 of the circumcircle of the regular polygon or star polygon is calculated with using the frequency f of the circumcircle wave and the sampling rate s. The frequency of the circumcircle wave is not necessarily the same as the frequency of the complementary n-gon waves, see also equation (7) and figure 10.

$$r_0 = s/4f \tag{2}$$

The variable k is used as index of the vertexes of the polygon the complementary n-gon waves are derived from.

$$0 \leq k < n \tag{3}$$

The differences Δx_k and Δy_k between the vertices k and $k+1$ are calculated for the x-axis and the y-axis in equations (4) and (5). ϕ is the initial phase that also rotates the reconstructed polygon the waves are derived from.

$$\Delta x_k = \cos(\theta(k+1) + \phi) - \cos(\theta k + \phi) \tag{4}$$

$$\Delta y_k = \sin(\theta(k+1) + \phi) - \sin(\theta k + \phi) \tag{5}$$

The variable m_k in equation (6) is the slope of the line that connects the vertices k and $k+1$ of the n-gon wave.

$$m_k = \Delta y_k / |\Delta x_k| \tag{6}$$

In equation (7) the full length of both waves T_n is calculated with a unit radius of 1. Both waves do have the same length.

$$T_n = \sum_{k=0}^{n-1} |\Delta x_k| \tag{7}$$

The radius of the circumcircle of the complementary n-gon waves is calculated in equation (8). The variables λ, ϵ and η are used to scale the radius and hence the frequency of the two waves. For a more detailed description about scales that can be derived from n-gon waves, see the paper *N-Gon Waves – Audio Applications of the Geometry of Regular Polygons in the Time Domain* [1]. Setting the exponent η to a value of -1 and the exponents λ and ϵ to a value of 0 will tune the frequency of the complementary n-gon waves to the frequency of the circumcircle wave. For tuning complementary n-gon waves see also equation (15).

$$r_n = \frac{r_0 \sec(\theta/2)^{\lambda} \csc(\theta/2)^{\epsilon}}{(4/T_n)^{\eta}} \tag{8}$$

The sum of the partial lengths T_k of the connected lines of the complementary n-gon waves are calculated in equation (9).

$$T_k = r_n \sum_{l=0}^{k-1} |\cos(\theta(l+1) + \phi) - \cos(\theta l + \phi)| \tag{9}$$

The variable μ_k in equation (10) is the step size or slope of the amplitude increments in the complementary angular wave.

$$\mu_k = \frac{-\Delta x_k}{r_0 |\Delta x_k|} \tag{10}$$

The variable t is the index of the sequence of samples of the complementary n-gon waves.

$$\lfloor T_k \rfloor \leq t < \lfloor T_{k+1} \rfloor \tag{11}$$

The calculation of the discrete n-gon wave is given in equation (12). The parameter a is the amplitude of the n-gon wave.

$$w_y(a, k, t, \phi, \theta) = a(\sin(\theta k + \phi) + (t - \lfloor T_k \rfloor) m_k) \tag{12}$$

The calculation of the discrete complementary angular wave is given in equation (13).

$$w_x(a,k,t,\phi,\theta) = a(\cos(\theta k+\phi) + (t - \lfloor T_k \rfloor)\mu_k) \quad (13)$$

Depending on the implementation of the equations, care should be taken to avoid divisions by zero. Equation (14) shows one way of how divisions by zero can be detected that are introduced by the initial phase when a digon with Schläfli symbol $\{2\}$, i.e. a line, approaches a vertical position and may become of infinitely short duration. If the result of the equation is zero, a division by zero was detected. Rounding errors may also be considered when dividing the initial phase ϕ by the central angle θ. If detected, a division by zero can be handled with giving a vertical slope a slight bias.

$$0 = 2([(\phi/\theta) \bmod n] + q) \bmod n \quad (14)$$

2.3. Tuning Complementary N-gon Waves to a Frequency

To tune the complementary n-gon waves to the same frequency as the circumcircle wave, the exponents in equation (8) can be set to the following values: $\lambda = \epsilon = 0$ and $\eta = -1$. Equation (8) for r_n can then be shortened to equation (15).

$$r_n = \frac{r_0}{(4/T_n)^{-1}} = s/fT_n \quad (15)$$

2.4. Timbral and Spatial Characteristics of Complementary N-gon Waves

Complementary n-gon waves offer a wide range of timbres, from some that are similar to the timbres of common sawtooth, triangle and square waves to buzz-like, coarse, percussive and click-like sounds. As they are angular waveforms they are rich in harmonics, but also produce aliasing. A more detailed description of the various timbres of these waveforms is beyond the scope of this paper.

The inherent spatial characteristics of complementary n-gon waves are audible on a stereophonic two-speaker system, where a separate audio channel and speaker is used for each of the two separate waves. For example, low frequency complementary n-gon waves derived from star polygons could be described as sounding similar to a percussive rotary audio effect: when the amplitude of the n-gon wave increases, the amplitude of the complementary angular wave decreases, as can be seen from figure 3. When this complementary n-gon wave reaches the part with the shortest durations between its peak amplitudes, a drum-like percussive sound is audible. As the n-gon wave reaches this part and it also reaches its maximum peak amplitudes, the n-gon wave's sharp drum-like sound is more prominent than the drum-like sound of the complementary angular wave. For an additional description of the drum-like sound see also the paper *N-Gon Waves – Audio Applications of the Geometry of Regular Polygons in the Time Domain* [1]. The drum-like sound of the complementary angular wave sounds softer and at its peak amplitudes the wave gives more room to its buzz-like sounds.

3. ORTHOGONAL COMPLEMENTARY N-GON WAVES

3.1. Orthogonal Complementary N-gon Waves Definition

If a Cartesian coordinate system is used, where the x-axis and the y-axis are perpendicular to one another, two orthogonally oriented complementary n-gon waves can be derived from the same regular polygon or regular star polygon. One can be derived by using the x-axis and another by using the y-axis for the amplitudes of the n-gon wave and the complementary angular wave. The initial phase of the complementary n-gon waves with amplitudes on the x-axis is calculated by adding the cosine of the central angle of the polygon to the initial phase of the complementary n-gon waves with amplitudes on the y-axis, which is shown in equation (16). If the variable ψ in the equation is incremented in integer values, it rotates the polygon in central angle steps and the value $n/4$ adds the cosine to this value.

$$\phi = \theta(\psi + n/4) \quad (16)$$

As orthogonal complementary n-gon waves consist of two pairs of complementary n-gon waves, a total of four angular waveforms can all be derived from one regular polygon or regular star polygon: two orthogonal n-gon waves and two orthogonal complementary angular waves, as illustrated in figure 4.

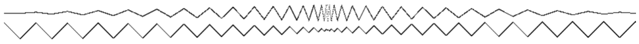

Figure 3: *Complementary n-gon waves derived from a star polygon with Schläfli symbol $\{128/63\}$. The wave on the top is the n-gon wave, the one on the bottom is the complementary angular wave. The image only shows excerpts of the waveforms.*

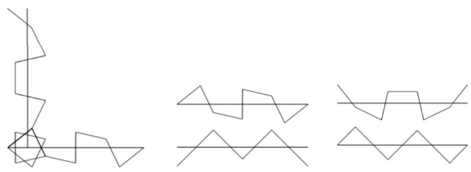

Figure 4: *Orthogonal n-gon waves with the perpendicular x-axis and y-axis and two complementary n-gon waves with the n-gon wave on top and the complementary angular wave on bottom. All derived from the same heptagram with Schläfli symbol $\{7/2\}$.*

3.2. Tuning Orthogonal Complementary N-gon Waves to a Frequency

If the initial phase ϕ is 0 and r_n is not calculated with equation (15), the orthogonal complementary n-gon waves do not have the same frequency. To tune orthogonal complementary n-gon waves to the same frequency, the initial phase can be set to $\pi/4$ rad and u multiples of $\pi/2$ rad added, as in equation (17) and illustrated in figure 5. If the waveforms should have the same frequency as the circumcircle wave, equation (15) can also be used to compute r_n.

$$\phi = \pi(1/4 + u/2), u \in \mathbb{N} \quad (17)$$

3.3. Two Channel Combinations of Orthogonal Complementary N-gon Waves

If two channels are used for audio or visual representation, orthogonal complementary n-gon waves offer 16 different waveform combinations derived from one regular polygon or star polygon. If

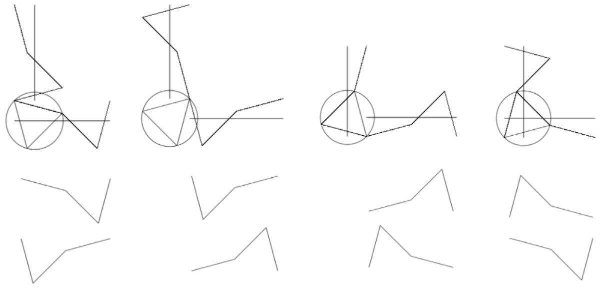

Figure 5: *Orthogonal n-gon waves derived from a trigon with Schläfli symbol $\{3\}$, tuned to the same frequency with computing the initial phase ϕ with equation (17). The perpendicular lines in the top row show the length and position of the n-gon waves. In the second row the n-gon waves are both shown with time on the horizontal axis and amplitude on the vertical axis.*

a digital or cathode-ray oscilloscope in X-Y mode is used to visualise the two audio channels, it can be seen that some of the angular shapes on the display of the oscilloscope appear to be the same for some combinations. However, if two independent audio channels are used with different locations in a stereophonic sound field, each combination sounds different. The figures 6 and 7 show orthogonal complementary n-gon waves derived from a dodecagram with Schläfli symbol $\{12/5\}$. Figure 8 visualises the X-Y mode shapes of the 16 combinations of the four waveforms of these orthogonal complementary n-gon waves.

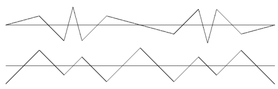

Figure 6: *N-gon wave and complementary angular wave with initial phase $\phi = 0$ rad. The Schläfli symbol of the polygon the waves are derived from is $\{12/5\}$.*

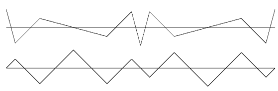

Figure 7: *N-gon wave and complementary angular wave with initial phase $\phi = \theta(n/4)$. The Schläfli symbol of the polygon the waves are derived from is $\{12/5\}$.*

3.4. Initial Phase and Two Channel Complementary N-gon Wave Combinations

If the initial phase of a two channel complementary n-gon wave combination is changed, a variety of waveforms and shapes can be generated. In addition, the initial phase of the individual waves can be changed independently. A more detailed description of two channel complementary n-gon wave combinations is beyond the scope of this paper. Some examples of possible combinations are illustrated in figure 9.

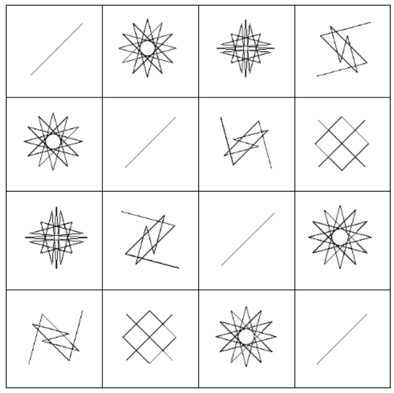

Figure 8: *Two channel combination matrix of geometric shapes constructed with orthogonal complementary n-gon waves. Example with orthogonal complementary n-gon waves derived from a dodecagram with Schläfli symbol $\{12/5\}$, ϕ = initial phase, f = frequency. Complementary n-gon waves with amplitudes on y-axis (y): $\phi = 0$, $f = 68.595016$ Hz. Complementary n-gon waves with amplitudes on x-axis (x): $\phi = \pi/n$, $f = 68.594955$ Hz. The order of the four vertical inputs to the matrix is the same as the horizontal: 1. n-gon wave (y), 2. complementary angular wave (y), 3. n-gon wave (x), 4. complementary angular wave (x). Although some of the visualisations look the same, the timbral and spatial characteristics of each combination are different.*

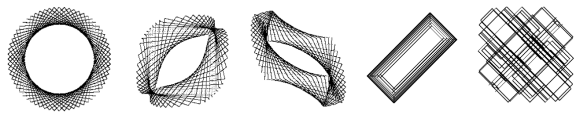

Figure 9: *Examples of two channel orthogonal complementary n-gon waves combinations. Description and numbering is from left to right. Four examples derived from a polygon with Schläfli Symbol $\{64/17\}$ and initial phase $7\pi/8$ rad: 1. complementary n-gon waves, 2. two orthogonal n-gon waves, 3. orthogonal n-gon wave and complementary angular wave, 4. two orthogonal complementary angular waves. One example derived from a polygon with Schläfli Symbol $\{64/31\}$ and initial phase $25\pi/32$ rad: 5. two orthogonal complementary angular waves.*

3.5. Timbral and Spatial Characteristics of Two Channel Orthogonal Complementary N-gon Waves Combinations

The timbres of two channel orthogonal complementary n-gon waves combinations are similar to the timbres of complementary n-gon waves, with the exception of two-part polyphony or phasing that occur when the waveforms are not tuned to the same frequency and some alternating rotary spatial characteristics. If not tuned to the same frequency with the tuning equations (15) and (17), the individual waveforms of two channel combinations of orthogonal complementary n-gon waves do not necessary have the same frequency. A result of this is the two-part polyphony or phasing. The two-part polyphony and the phasing are dependent on the frequency of the unit circle wave, the initial phase and the number of vertices of the polygon of which the waveforms are derived from. For instance, if the initial Schläfli symbol $\{3\}$ of a regular polygon of which orthogonal n-gon waves with different frequencies are derived from is successively increased, at some point the two-part polyphony converges into phasing. The closer these orthogonal n-gon waves with different frequencies get to an n-gon wave derived from an apairogon (a regular polygon with an infinite number of sides with Schläfli symbol $\{\infty\}$) and the closer the n-gon wave gets to a circle wave, the less phasing occurs. This is due to the fact that in this case the difference between the initial phases and hence the difference between the frequencies of the orthogonal n-gon waves theoretically become infinitely small. Figure 10 shows an approximation of orthogonal n-gon waves to a circle wave. Although, phasing can be a desired effect used for audio and visual synthesis, it can also occur as an unintended effect produced by rounding and truncation errors during the computation of the waveforms. If orthogonal complementary n-gon waves derived from the same regular polygon or star polygon are tuned to the same frequency with equation (15) or (17), the amplitude peaks of the two audio channels of the percussive rotary audio effect described above in "2.4 Timbral and Spatial Characteristics of Complementary N-gon Waves" are alternating. The tempo of the percussive sound appears doubled, because the amplitude peaks change periodically from one channel to the other channel, as illustrated in figure 11.

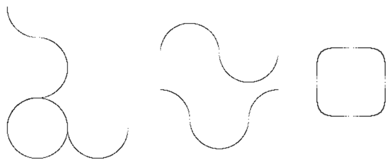

Figure 10: *Orthogonal n-gon waves derived from a regular polygon with Schläfli symbol $\{64\}$ and the resulting shape. Although, the orthogonal n-gon waves get close to a circumcircle wave, which is a circle wave derived from a circumcircle, phasing is still audible.*

4. SHUFFLED SAMPLES NOISE APPLIED TO COMPLEMENTARY N-GON WAVES

4.1. Applying a Fisher-Yates Shuffle Algorithm to Complementary N-gon Waves Samples

A shuffle algorithm can be used to permute the sequence of samples of complementary n-gon waves or any other digital signal. As the shuffling often produces noise-like signals, this effect was given the name shuffled samples noise. In the software that was used to explore the application of the shuffled samples noise on complementary n-gon waves and to create the images in this paper, a Fisher-Yates shuffle algorithm [2] was used to shuffle sequences of samples. The Fisher-Yates shuffle algorithm was selected because it creates random permutations of an initial sequence and any of its permutations has the same probability of occurrence. It needs to be mentioned that a bias can be introduced by random number generators and modulo operations used in software implementations of the Fisher-Yates shuffle algorithm. But this is not discussed in this paper. Two approaches how a Fisher-Yates shuffle algorithm was applied to complementary n-gon waves are presented in this paper, shuffling a complete sequence and shuffling subsequences of a sequence.

A digitised two-dimensional polygon is composed of a set of samples of points stored in a sequence on a digital data storage device. If the x- and y-positions of the points of the polygon were calculated and stored in the same sequence before they are visualised, any permutation of the sequence of the x- and y-positions will visualise the polygon on a digital display device, as can be seen from figure 12. This also applies to complementary n-gon waves, as complementary n-gon waves are derived from a two-dimensional digitised polygon and the x- and y-positions of the points of the polygon are stored in the sequences of the samples of the n-gon wave and the complementary angular wave. These two sequences of samples of the complementary n-gon waves are of same size and can be processed simultaneously as if they would be a single sequence of samples of points. Hence, this single sequence of the samples of complementary n-gon waves can be permuted by shuffling it with the Fisher-Yates shuffling algorithm and the samples can therefore be rearranged into noise signals, while the initial polygon can still be visualised on a digital display device. The number of permutations p_w of the sequence of samples of complementary n-gon waves can be calculated with equation (18).

$$p_w = \left(\sum_{k=0}^{n-1} |r_n \Delta x_k|\right)! = (r_n T_n)! \tag{18}$$

A cathode-ray oscilloscope in X-Y mode equipped with a Z input to control the electron beam intensity offers similar features as a digital display for visualising initial polygons. On a cathode-ray oscilloscope in X-Y mode not equipped with a Z input to control the intensity of the electron beam and depending on the order of the shuffled sequence of the samples of the complementary n-gon waves, the two digital signals seem to rarely visualise the initial polygon and often visualise fuzzy cloud-like distorted versions of the initial polygon, as illustrated in the second row of figure 13. The distortion of the visualisation of the initial polygon on a cathode-ray oscilloscope of this type can be reduced by shuffling only the subsequences of points forming the lines that connect the vertices of a polygon, but not shuffling the complete sequence. The

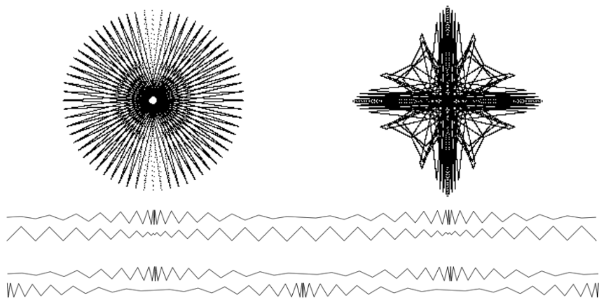

Figure 11: *Visualisations of reconstructed shapes of complementary n-gon waves on the left and orthogonal n-gon waves on the right, both derived from a polygon with Schläfli symbol $\{64/31\}$. The shapes are visualised with points that represent the amplitudes of the samples. It can be seen that the steeper a slope of a line is, the less samples are used to build the line. The first pair of waveforms under the shapes was used to construct the shape on the left and the second pair of waveforms was used to construct the shape on the right. If a low frequency is used for the waveforms, the alternating two-channel amplitude patterns of the waveforms produce percussive-like sounds with a steady beat. The second pair of waveforms appears to double the tempo of the beat. This is due to the initial phase difference of the orthogonal n-gon waves.*

first and last points of the subsequences need not to be shuffled, but all the other points within the subsequences are shuffled. This will reduce the distortion a bit more, as the vertices of the visualised polygon are built with these points and kept in place, as illustrated in the third row of figure 13. The number of permutations p_l for the shuffled subsequences of samples that form the lines of complementary n-gon waves can be calculated with equation (19).

$$p_l = \prod_{k=1}^{n} (|r_n \Delta x_k| - 2)! \tag{19}$$

The resulting sequences with their shuffled subsequences are also permutations of the complete initial sequence. However, when the whole initial sequence is shuffled, the probability of the occurrence of these permutations decreases proportionally to the number of subsequences increased and to the size of the complete sequence increased. The probability $P(S)$ of the occurrence of shuffled subsequences of samples that form the lines within the whole sequence of samples of complementary n-gon waves can be calculated with equation (20).

$$P(S) = p_l / p_w \tag{20}$$

The initial unshuffled sequence of samples of complementary n-gon waves is also one possible permutation of its shuffled samples. The probability $P(N)$ that it occurs when all samples are shuffled can be calculated with equation (21) and when the subsequences of the samples of the lines are shuffled with equation (22).

$$P(N) = 1/p_w \tag{21}$$

$$P(N) = 1/p_l \tag{22}$$

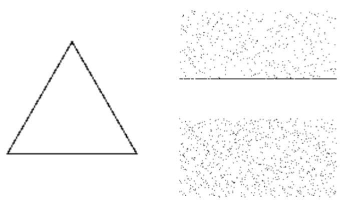

Figure 12: *If points instead of lines are used to visualise digital samples and the effect of shuffled samples noise applied to the whole sequence of samples of the complementary n-gon waves derived from a trigon, the trigon is not filled as shown in the second row of figure 13, but is constructed as a set of points of the lines of the trigon. If a third wave would be used to control the intensity of the beam of a cathode-ray oscilloscope in X-Y mode with a Z input, a similar effect may be achieved.*

4.2. Timbral and Spatial Characteristics of Shuffled Samples Noise Applied to Complementary N-gon Waves

Shuffled samples noise could be regarded as an audio effect applied to digital audio signals. The two approaches described above that can be used to create shuffled samples noise signals from complementary n-gon waves produce different audio effects, which to some degree are analogous to the visualisations on a cathode-ray oscilloscope in X-Y mode. Shuffling the complete sequence of complementary n-gon waves with low frequencies often produces

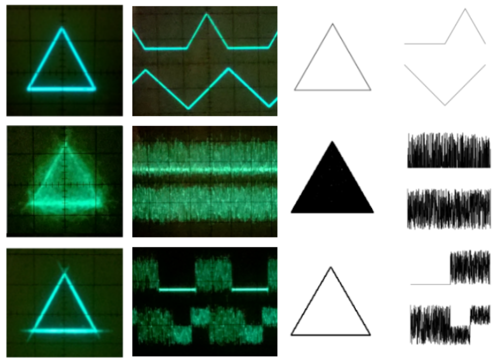

Figure 13: *Examples of shuffled samples noise applied to complementary n-gon waves derived from a trigon, a regular polygon with Schläfli symbol* $\{3\}$*. The pictures of the display of a cathode-ray oscilloscope in X-Y mode on the left show visualisations of the digital complementary n-gon waves on the right. The first row shows complementary n-gon waves and the visualisations of the trigon they were derived from. On the second row, the effect of applying shuffled samples noise to the whole sequence of samples of the complementary waves on the first row can be seen. The third row shows the effect of applying shuffled samples noise to the subsequences of the samples of the lines of the complementary n-gon waves on the first row.*

a variety of noise signals, sometimes sounding similar to common coloured noise signals. Increasing the frequency turns the noise into buzz-like sounds. Shuffling only the samples of the points of the lines that connect the vertices of complementary n-gon waves also generates frequency related noise and buzz-like sounds, but the changing amplitude levels of the connecting lines introduce spatial and rhythmic characteristics similar to complementary n-gon waves. The number of vertices, the initial phase and polygon density of the polygon the waveforms are derived from as well as the frequency of the waves are all parameters that produce a range of different buzz-like sounds and noise signals. This is mainly due to the number of samples that are used to construct the slopes of the lines of complementary n-gon waves. The steeper the slope of a line, the less samples are used to form a line, as illustrated in figure 14.In addition, when shuffling only the samples of the points of the lines and the closer the regular polygon of which the complementary n-gon waves are derived from is to an apairogon (i.e. a regular polygon with Schläfli symbol $\{\infty\}$), the less noise is produced because the lines get shorter and there are less samples left to be shuffled. In this case, the amount of noise in the shuffled samples noise signal is related to the shape of the initial polygon. Figures 14 and 15 show examples of how different parameter settings of complementary n-gon waves can produce a variety of noise signals.

5. ORTHOGONAL COMPLEMENTARY N-GON WAVES OSCILLATOR PROTOTYPE SOFTWARE

A prototype of an orthogonal complementary n-gon waves oscillator was developed by the author as a VST-plugin in the programming languages C and C++ in 2019 and 2020 using the JUCE framework [14]. As it is an orthogonal complementary n-gon waves oscillator, it generates four waveforms. The shuffled samples noise

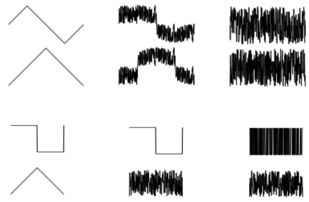

Figure 14: *Shuffled samples noise applied to complementary n-gon waves derived from a tetragon, a polygon with Schläfli symbol* $\{4\}$*. The first row and second row visualise complementary n-gon waves with different initial phases. The first column shows the complementary n-gon waves. The second and third columns show shuffled samples noise applied to these waves, with the second showing the shuffled subsequences of lines approach and the third the complete sequence shuffled approach.*

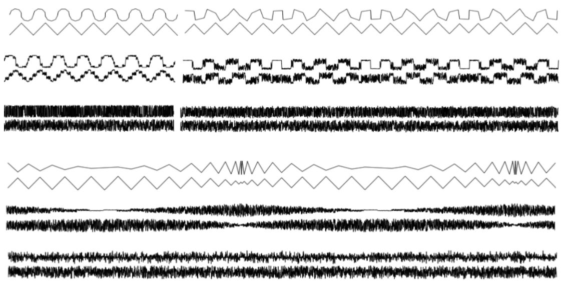

Figure 15: *Shuffled sample noise applied to complementary n-gon waves derived from polygons with Schläfli symbols:* $\{64/7\}$*,* $\{64/17\}$*,* $\{64/31\}$*, clockwise starting from top left. This illustrates the effect of the polygon density on shuffled samples noise signals. First rows show the complementary n-gon waves, second rows shuffled samples noise signals produced with shuffling the subsequences of the samples of the lines and third rows shuffled samples noise signals with the complete sequence of samples shuffled.*

effect is included and can be applied to the waveforms. The prototype features a graphical user interface with controllers for the parameters of the waveforms and a digital display for visualisations of the waveforms. The two-channel audio outputs of the laptop (HP 350 G2, i5-5200U, 2.2 GHz, 16 GB RAM, Windows) that ran the prototype were connected to the X and Y inputs of a cathode-ray oscilloscope via an USB audio interface. The complexity of the complementary n-gon waves function implementation in the programming language C was roughly $O(n+T_n r_n)$ and the complexity of the Fisher-Yates shuffle algorithm implementation was roughly $O(n)$. On this laptop the computation of orthogonal complementary n-gon waves with a sampling rate of 48 kHz and a frequency of 1 Hz took about 0.000109 seconds and for a frequency of 440 Hz about 0.000008 seconds. Equation (14) was used for the error handling of divisions by zero. The prototype VST-plugin was used to test and explore the geometry and audiovisual characteristics of orthogonal complementary n-gon waves and shuffled samples noise. The audiovisual characteristics of these waveforms described in this paper are results of experiments with the prototype. All figures in this paper are screenshots of the prototype's digital display or pictures of the visualisations of its audio output on a cathode-ray oscilloscope. Experiments not documented in this paper included other complementary n-gon waves and shuffled samples noise combinations, MIDI control of the waveform parameters and audiovisual cross-fading.

6. EVALUATION AND DISCUSSION

Two n-gon waves can be used to visualise shapes in a similar way as sine waves can be used to visualise Lissajous Figures. A range of geometric shapes can be visualised with changing the parameter values of the polygons the n-gon waves are derived from (for example, Schläfli symbol, polygon rotation) and of the waveforms themselves (for example, frequency, amplitude, phase offset). However, it was found that two n-gon waves cannot reconstruct or visualise the polygon they were derived from, with the exception of n-gon waves derived from a tetragon (i.e. a square). As shown in this paper, the reconstruction of a polygon with an n-gon wave requires an additional wave derived from the same polygon as the n-gon wave: the complementary angular wave. This waveform with orthogonal angles is a tetragon wave when derived from a regular polygon, but it is not an n-gon wave when derived from a star polygon. With the exception of the calculation of the slopes of the waveforms and using sines for the n-gon waves and cosines for the complementary angular wave, the equations for the waveforms are the same. As the two waveforms complement each other and are both derived from the same polygon with similar mathematics, they were named complementary n-gon waves. It could be debated whether the relation of the two waveforms is evident from the names of the waveforms given to them in this paper.

The usage of two Cartesian coordinates axes, the x-axis and y-axis, for the time and amplitude values of complementary n-gon waves for audiovisuals led to further explorations in this domain with orthogonal complementary n-gon waves. It was found that combinations of orthogonal complementary n-gon waves, which are oriented orthogonally along the Cartesian axes, enhance the audiovisual range of complementary n-gon waves. For instance, in the audio domain with phasing and two-part polyphony and in the visual domain with a range of geometric shapes. Orthogonal complementary n-gon waves can be tuned to the same frequency with polygon rotation. Hence, four waveforms with the same frequency can be derived from the same polygon. Whether this feature could be described as a single oscillator that outputs four waveforms or as four separate oscillators needs further investigations.

The study of polygons in the audiovisual domain led to the observation of an audiovisual phenomenon: the same or a very similar polygon can be visualised with different types of audible waveforms. To investigate this audiovisual phenomenon the shuffle algorithm by Fisher-Yates was applied to the samples of orthogonal complementary n-gon waves. This resulted in digital audio signals that ranged from buzz-like waveforms to noise signals. They were used to geometrically reconstruct and visualise the initial polygon on digital display devices and within limitations also on a cathode-ray oscilloscope in X-Y mode. It was found that the shuffle algorithm produced a characteristic audio effect.

An orthogonal complementary n-gon waves oscillator prototype was developed. Tests with the prototype have shown that, depending on the computation time, orthogonal complementary n-gon waves and the shuffled samples noise effect could be used for realtime or wavetable audio and visual synthesis. The computation time of the waveforms depends on the implementation, hardware, sampling rate and frequency of the waveforms.

While developing the orthogonal complementary n-gon waves oscillator prototype considerations emerged about how the geometry of the waveforms and the polygons of which the waveforms were derived from could be visualised with matching human audio and visual perception. The considerations may also apply to audiovisual software development in general, which is beyond the scope of this paper. However, two considerations that were found to be fundamental are briefly discussed. One consideration is whether to use lines that connect the point samples of geometric shapes stored in the waveforms for oscilloscope-like visualisations or to use disconnected point samples for geometry-like visualisations. Note that in geometry a circle can be defined as a set of points with equal distance to a given point. As illustrated in this paper, the orthogonal complementary n-gon waves oscillator prototype includes both point and line visualisations, as it was found useful for creative visualisations and experimentation. Another fundamental consideration is concerned with visual aliasing, i.e. static or rotating visualisations of geometric shapes and dynamic oscilloscope-like waveform visualisations. A zoomable static visualisation, which only changes when a geometric parameter is changed, can provide a clear geometric representation of the waveforms and shapes without visual aliasing. For creative and further research applications a flexible approach could be used: An audiovisual software could include the option to change the visualisation from a static display of the waveforms and shapes to a dynamic display that visualises the data read from the audio buffer. The considerations on how audiovisual software can match human audiovisual perception of geometric shapes are being investigated in the author's current research.

7. CONCLUSIONS

In this paper complementary n-gon waves and the shuffled samples noise effect were introduced with equations and their timbral and spatial characteristics described. It was shown how the Fisher-Yates shuffle algorithm can be applied to complementary n-gon waves with shuffling the complete sequence or subsequences of the samples of the waveforms. Depending on the computation time, it was found that complementary n-gon waves and shuffled samples noise could be used for realtime or wavetable audio and visual synthesis. Considerations concerning audiovisual perception and the development of audiovisual software applications were briefly discussed and may need further investigations. A prototype of an orthogonal complementary n-gon waves oscillator VST-plugin was developed. The prototype also included the shuffled samples noise effect and was used as a tool to investigate the relations of geometry, audio, visual and perception. An audiovisual phenomenon was observed: a polygon or a very similar shape can be visualised with different types of waveforms with different timbral and spatial characteristics. If shuffled samples noise was applied to complementary n-gon waves, it was still possible to visualise polygons on a digital display device and, within limitations, on a cathode-ray oscilloscope in X-Y mode without a Z input for controlling the intensity of the electron beam. According to the observations made, it could be concluded at this point that it can be difficult to tell by ear or from visualised shapes, which specific types of waveforms were used to visualise what specific types of shapes. A person trained in this domain of audiovisual geometry may be able to match some types of audible waveforms with visualised shapes and vice versa. For example, complementary n-gon waves derived from star polygons can have a distinct drum-like sound. Also, if subsequences of the samples of these waves are shuffled, characteristic noise patterns can be produced. However, if the complete sequences of the samples of these waves are shuffled, it may be challenging for the same person to find a match. This audiovisual phenomenon is being further investigated in the author's current research on audiovisual perception of geometric shapes and further results are planned to be presented in a future publication.

8. ACKNOWLEDGMENTS

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